Hydraulic & Pneumatic Actuators

- References
- Introduction
- Fluid System Fundamentals
- Electrohydraulic Valve-Controlled Servomechanism Case Study
- Pneumatic System Closed-Loop, Computer-Controlled Positioning Experiment and Case Study

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Introduction

- Applications of Hydraulic and Pneumatic Actuators
- Hydraulic / Pneumatic Systems vs. Electromechanical Systems
- Hydraulic Systems vs. Pneumatic Systems

Applications of Hydraulic & Pneumatic Actuators

- *Hydraulic and Pneumatic Control System* components include pumps, pressure regulators, control valves, actuators, and servo-controls.
- *Industrial Applications* include automation, logic and sequence control, holding fixtures, and high-power motion control.
- *Automotive Applications* include power steering, power brakes, hydraulic brakes, and ventilation controls.
- *Aerospace Applications* include flight-control systems, steering-control systems, air conditioning, and brake-control systems.

Hydraulic / Pneumatic Systems vs. Electromechanical Systems

- Power Density Capability
  - Electromagnetic motors, generators, and actuators are limited by magnetic field saturation and can produce up to about 200 pounds per square inch of actuator.
  - In hydraulic systems, 3000 to 8000 pounds per square inch of actuator is common in aircraft applications and 1000 pounds per square inch is common in industrial applications.
  - Hydraulic systems, both actuators and generators, benefit from a greater ratio of force per unit volume.

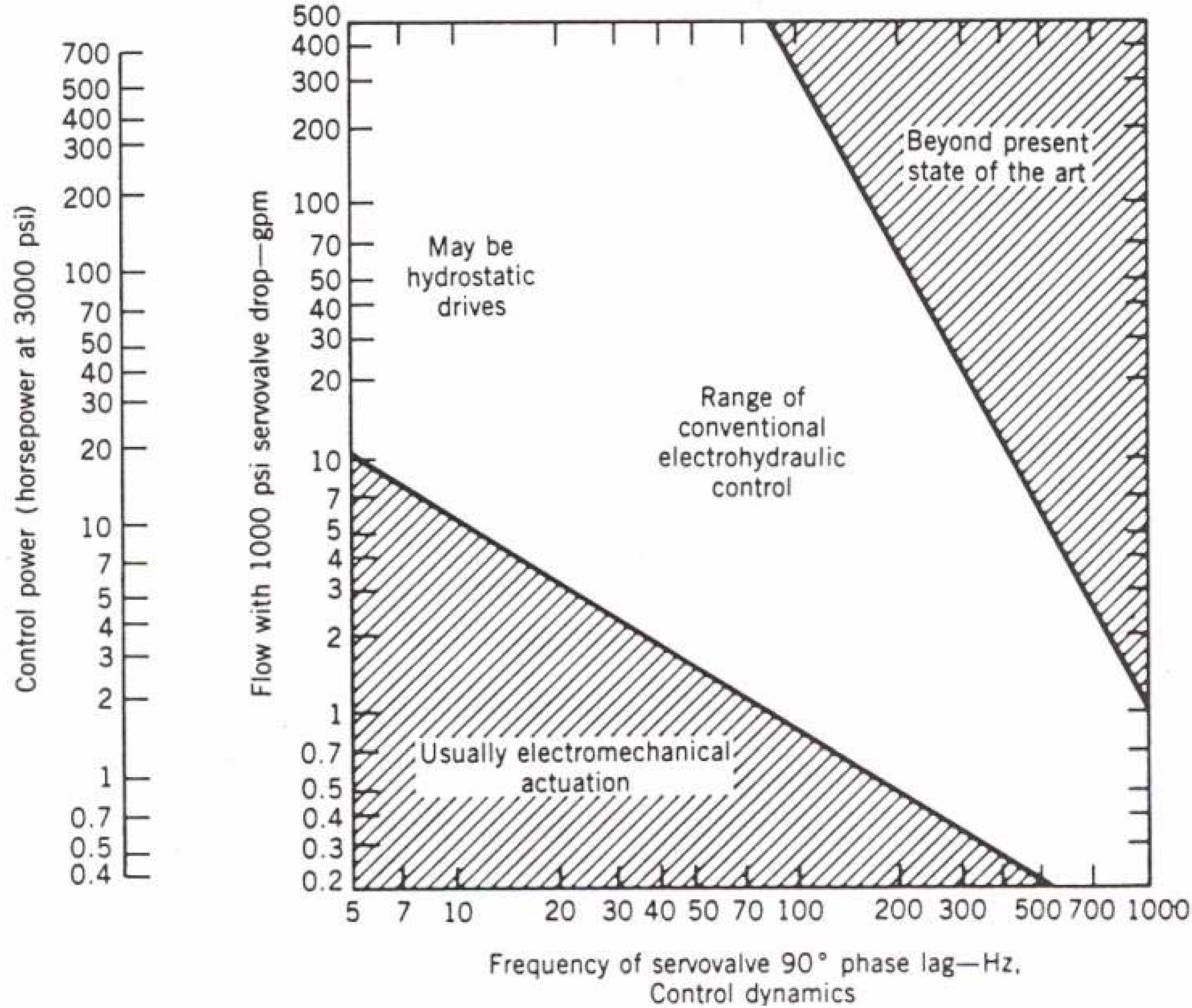
- Responsiveness and Bandwidth of Operation
  - Electromagnetic actuators have a large inertia associated with their motion, so they cannot accelerate quickly.
  - Hydraulic and pneumatic systems are more responsive and have a greater bandwidth of operation at the same power output levels.
- Heat Dissipation
  - Fluid circulating to and from an actuator removes heat generated by the actuator that is doing work.
  - Electromechanical actuators and motors have limited ability to dissipate heat generated inside the device and rely on free or forced convection to the surrounding environment.

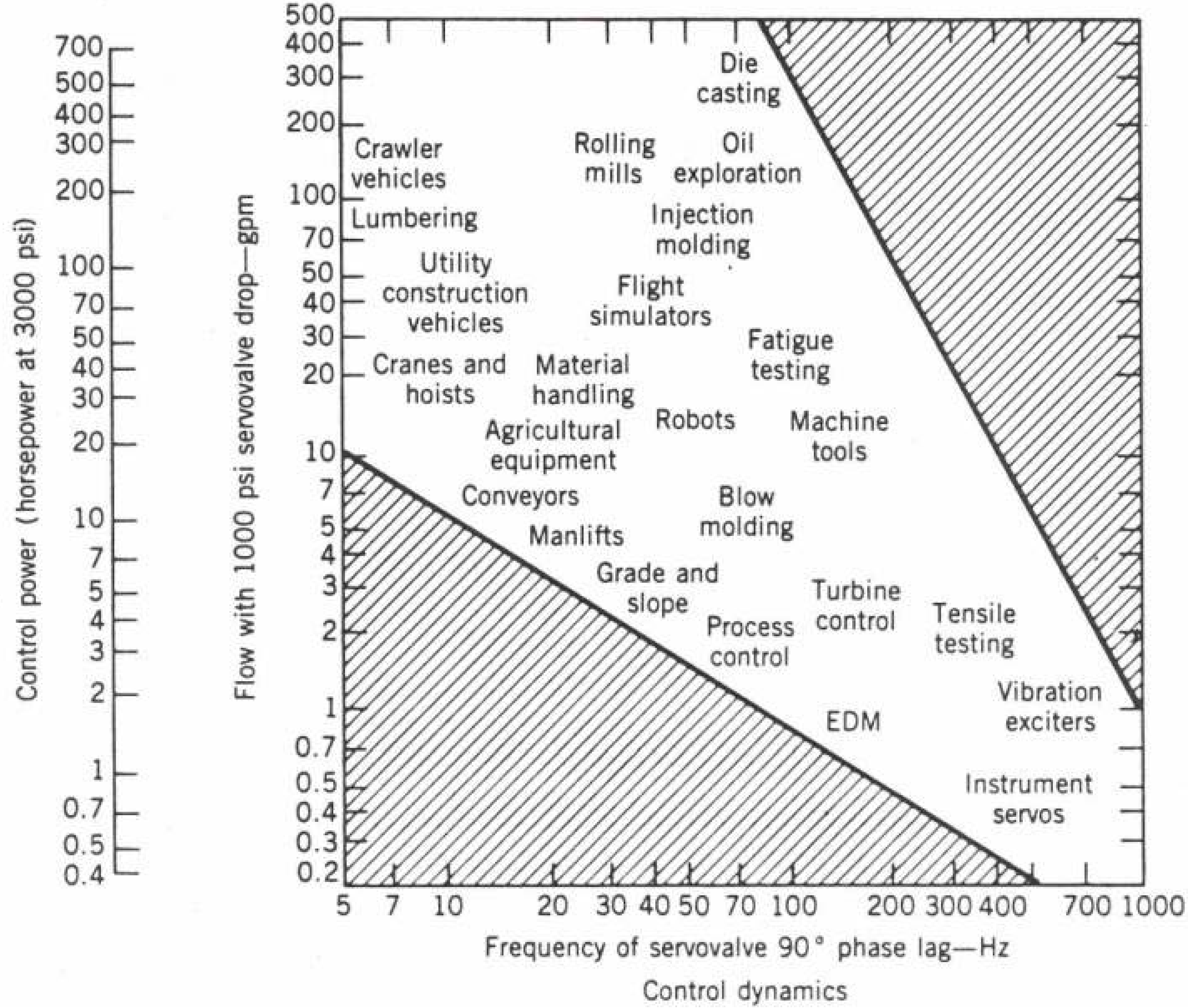
- Heat is the predominant damaging mechanism in electric and electronic systems.
- Reliability of electromagnetic devices is limited compared to that of hydraulic and pneumatic devices.

- <u>Modeling and Simulation</u>
  - Hydraulic and pneumatic systems generally have more significant nonlinearities than do electric or mechanical systems.
- <u>Miscellaneous</u>
  - Electric power is more readily available, cleaner and quieter, and easier to transmit, but may create electrical interference with low-level data signals and can cause overheating problems at low speeds.

– Hydraulic drives tend to be stiffer with respect to load disturbances; intermittent and stalled operation without damage is possible; but oil leakage, flammability, and fluid contamination may pose problems.

When both high speed and high power are required, hydraulic systems my be mandatory.





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Flow with 1000 psi servovalve drop—gpm
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Die casting
Crawler vehicles
Rolling mills
Oil exploration
Lumbering
Injection molding
Utility construction vehicles
Flight simulators
Fatigue testing
Cranes and hoists
Material handling
Robots
Machine tools
Agricultural equipment
Conveyors
Blow molding
Manlifts
Grade and slope
Turbine control
Tensile testing
Process control
Vibration exciters
EDM
Instrument servos
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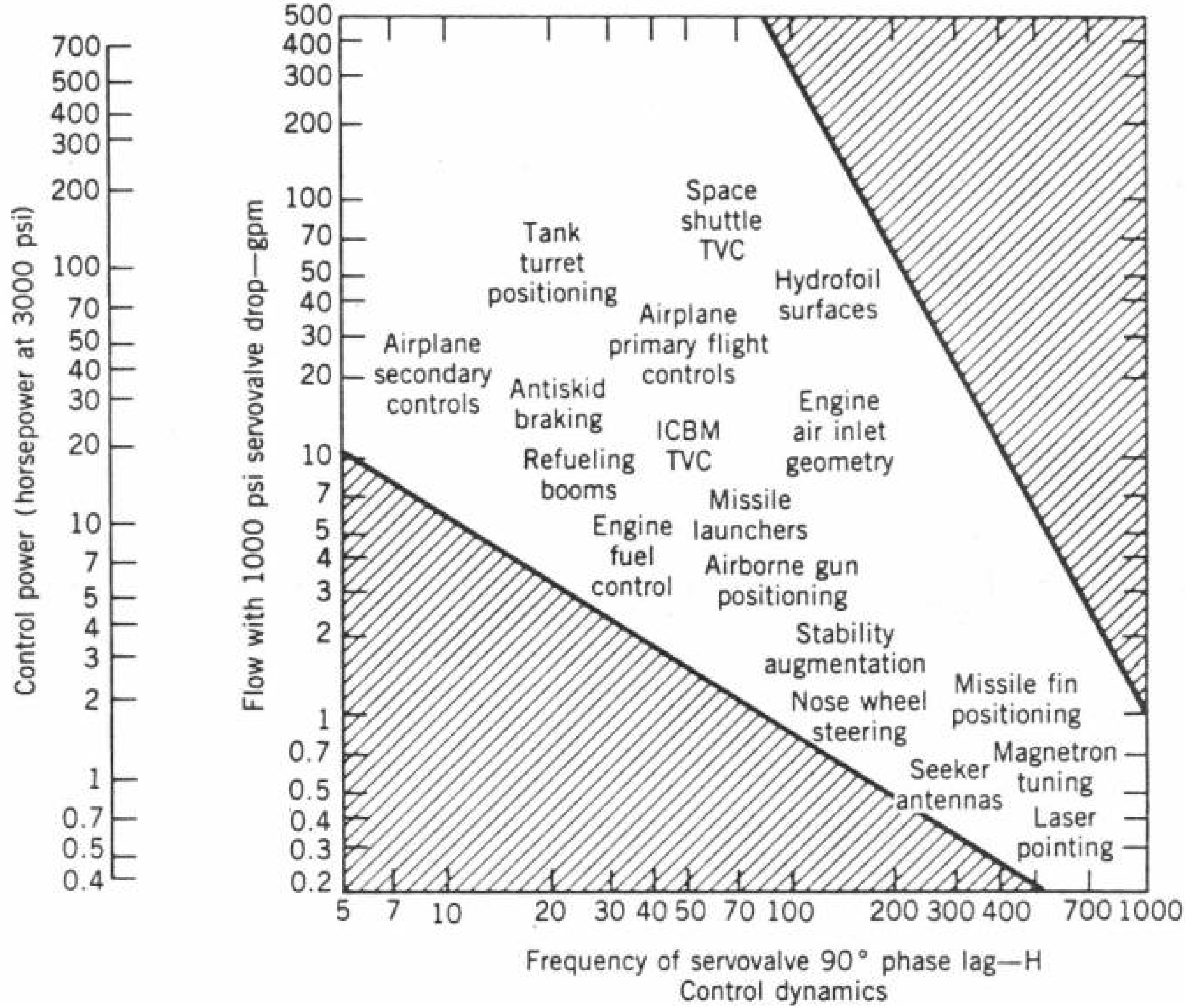
Frequency of servovalve 90° phase lag—Hz
Control dynamics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Space shuttle TVC
Tank turret positioning
Hydrofoil surfaces
Airplane primary flight controls
Airplane secondary controls
Antiskid braking
Engine air inlet geometry
ICBM TVC
Refueling booms
Missile launchers
Engine fuel control
Airborne gun positioning
Stability augmentation
Missile fin positioning
Nose wheel steering
Magnetron tuning
Seeker antennas
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Frequency of servovalve 90° phase lag—H
Control dynamics

Hydraulic Systems vs. Pneumatic Systems

- Compressibility of Fluid
  - Speed of response: hydraulic systems have a rapid initial response while pneumatic systems exhibit a time delay.
  - Stiffness to external load disturbances: pneumatic systems exhibit a lack of stiffness, especially to external load disturbances.
- Efficiency
  - The efficiency of pneumatic systems is much lower than that of hydraulic systems. Losses are caused by:
    - Cooling of the fluid which dissipates energy in the form of heat.

- Use of positive-displacement actuators causes loss of energy by expansion.
- Leakage – it is difficult to achieve satisfactory sealing. This also puts a limit on system pressure.
- Lubrication – The poor lubricating properties of gases compared to oils also leads to energy losses.

- High-Temperature Applications
  - The pneumatic system can work above 500°C where both electrical and hydraulic systems would fail.
- Mathematical Model Complexity
  - The basic equations of pneumatic systems are nonlinear and tend to be more complex than those of incompressible flow systems.

Fluid System Fundamentals

- Classification of Materials
- Fundamental Concepts & Properties of Fluids
- Basic Equations of Fluid Dynamics
- Lumped-Parameter Approach
- Passive Elements
  - Fluid Resistance
  - Fluid Capacitance
  - Fluid Inertance
- Lumped vs. Distributed Fluid System Models

- Fluid Impedance
- Fluid Sources: Pressure and Flow Rate

Classification of Materials

- Materials may be classified *rheologically* with respect to their shear stress – shear strain behavior in simple shear.
- *Rheology* is the science of the deformation and flow of matter.
- In general, a fluid will undergo a continuous deformation without rupture when subjected to a constant anisotropic stress, where as a solid will generally assume a static equilibrium configuration under such conditions.

- An *isotropic* quantity is the same in any direction from a given point in a system. *Anisotropy* implies a dependence on direction or orientation.
- This type of behavior is relative and depends upon the characteristic time required for the material to respond to a change in stress or strain relative to the time scale of observation, as well as the magnitude of the stress or strain.
- What is silly putty? What is granite?
  - Silly putty will fracture cleanly as a solid if subjected to a large suddenly-applied stress, while it will flow freely as a liquid when subjected to a constant stress of low or moderate magnitude.

- Granite, normally considered a solid, will flow measurably in large formations over a period of geologic time under the influence of gravity alone.

- Basic axiom of rheology is that "everything flows."

- <u>Classification of Materials</u>: Idealized Models
  - Spectrum of Material Classification in Simple Shear Deformations

Solid

Fluid

- Rigid Solid $\gamma = 0$
- Linear Elastic Solid (Hookean) $\tau = G\gamma \quad (G = \text{constant})$
- Nonlinear Elastic Solid $\tau = G(\lambda)\lambda$
- Viscoelastic $\tau = f(\gamma, \dot{\gamma}, t, \ldots)$
- Nonlinear Viscous Fluid (Non-Newtonian) $\tau = \eta(\dot{\gamma})\dot{\gamma}$
- Linear Viscous Fluid (Newtonian) $\tau = \mu\dot{\gamma} \quad (\mu = \text{constant})$
- Inviscid Fluid $\tau = 0$

$\gamma = \text{shear strain}$

$\dot{\gamma} = \text{shear rate}$

$\tau = \text{shear stress}$

Fundamental Concepts & Properties of Fluids

- Basic Concepts
- Definition of a Fluid: Liquids and Gases
- Density
- Equation of State: Liquids and Gases
- Viscosity
- Propagation Speed
- Thermal Properties
- Reynolds Number Effects
- Classification of Fluid Motions

Basic Concepts

- Continuum
  - Fluid is a continuum, an infinitely-divisible substance. As a consequence, each fluid property is assumed to have a definite value at each point in space. Fluid properties are considered to be functions of position and time, e.g., density scalar field $\rho = \rho\,(x, y, z, t)$ and velocity vector field $\vec{v} = \vec{v}(x, y, z, t)$.
- Velocity Field $\vec{v} = \vec{v}(x, y, z, t)$
  - Steady flow – all properties remain constant with time at each point.

$$\frac{\partial \eta}{\partial t} = 0 \quad \eta \text{ is any fluid property}$$

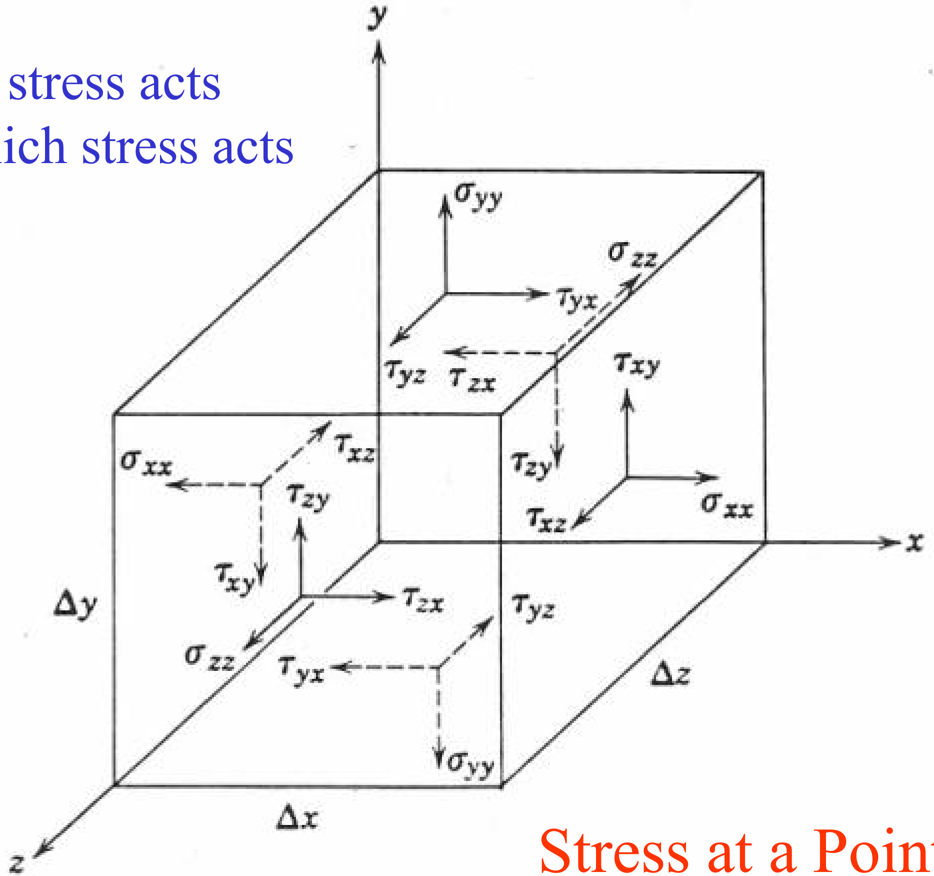
– One-, two-, three-dimensional flows – depends on the number of space coordinates required to specify the velocity field.
– Uniform flow – velocity is constant across any cross section normal to the flow. Other properties may be assumed uniform at a section.
– Timelines, pathlines, streaklines, and streamlines provide visual representation of a flow. In steady flow, pathlines, streaklines, and streamlines coincide.
– All fluids satisfying the continuum assumption must have zero relative velocity at a solid surface (no-slip condition) and so most flows are inherently two or three dimensional. For many problems in engineering, a one-dimensional analysis is adequate to provide approximate solutions of engineering accuracy.

- Stress Field
  - Surface forces: all forces acting on the boundaries of a medium through direct contact.
  - Body forces: forces developed without physical contact and distributed over the volume of the fluid, e.g., electromagnetic and gravitational forces.
  - Stresses in a medium result from forces acting on some portion of the medium. The concept of stress provides a convenient means to describe the manner in which forces acting on the boundaries of the medium are transmitted through a medium.
  - State of stress at a point can be described completely by specifying the stresses acting on three mutually perpendicular planes through the point.

– A stress component is considered positive when the direction of the stress component and the plane on which it acts are both positive or both negative.

1st subscript: plane on which stress acts
2nd subscript: direction in which stress acts

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



Notation for Stress

Stress at a Point

Definition of a Fluid

- A fluid is a substance (a continuum) that deforms *continuously* under the application of a shear stress no matter how small the shear stress may be and includes both liquids and gases. A fluid cannot sustain a shear stress when at rest.
- Liquids are nearly incompressible.
- Gases are highly compressible.
- Liquids are distinguished from gases by orders of magnitude differences in their density, absolute viscosity, and bulk modulus.

Density

- The density ρ of a fluid is defined as the mass m per unit volume V under specified conditions of pressure P_0 and temperature T_0:

$$\rho = \left.\frac{m}{V}\right|_{P_0, T_0}$$

Equation of State: Liquids

- An equation of state is used to relate the density, pressure, and temperature of a fluid.
- A relation derived from a Taylor series expansion is valid over limited ranges of pressure and temperature:

$$\rho(P,T) = \rho\left(P_0,T_0\right) + \left.\frac{\partial \rho}{\partial P}\right|_{P_0,T_0}\left(P - P_0\right) + \left.\frac{\partial \rho}{\partial T}\right|_{P_0,T_0}\left(T - T_0\right)$$

$$= \rho_0\left[1 + \frac{1}{\beta}\left(P - P_0\right) - \alpha\left(T - T_0\right)\right]$$

- $\beta \Rightarrow$ Bulk Modulus (inverse of compressibility)

$$\beta = \rho_0 \left.\frac{\partial P}{\partial \rho}\right|_{P_0, T_0} = \left.\frac{\partial P}{\partial \rho / \rho_0}\right|_{P_0, T_0}$$

  – The bulk modulus can be measured by noting the change in pressure with a fractional change in volume of a fixed mass of fluid:

$$\beta = -\frac{\partial P}{\partial V / V_0}$$

  – Isothermal Bulk Modulus (or merely bulk modulus) β can be used when the pressure changes occur at slow enough rates during heat transfer to maintain constant temperature.

- Adiabatic Bulk Modulus β_a can be used when the rate of pressure change is rapid enough to prevent significant heat transfer.
- C_p/C_v, the ratio of specific heats, is only slightly greater than 1.0 for liquids.

$$\beta = -\frac{\partial P}{\partial V / V_0} \qquad \beta_a = \frac{C_p}{C_v}\beta$$

- $\alpha \Rightarrow$ Thermal Expansion Coefficient

$$\alpha = -\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_{P_0, T_0}$$

  - Thermal Expansion Coefficient α relates the incremental change in volume with changes in temperature and can be expressed in terms of volumes for a fixed mass of fluid:

$$\alpha = \left.\frac{\partial V / V_0}{\partial T}\right|_{P_0, T_0}$$

$\alpha = 0.5 \times 10^{-3}/°F$ for most liquids

- The compressibility of power-transmitting fluids is a major factor in determining systems resonant frequency. This in turn puts limitations on the speed of response of the output.
- The actual value of the bulk modulus of oil is strongly dependent on the amount of air present in the form of bubbles. Some level of entrained air is impossible to avoid even in the best circuit designs.
- An additional factor which will further reduce the effective bulk modulus will be the type of pipework used to contain the oil.

Equation of State: Gases

- The equation of state for an ideal gas is

$$\rho = \frac{P}{RT}$$

- P and T are in absolute terms and R is the gas constant. Most gases follow this ideal behavior for considerable ranges of pressures and temperatures.

- A gas undergoing a polytropic process follows the relationship: $$\frac{P}{\rho^n} = C = \text{ constant}$$
  - n = 1.0 for an isothermal or very slow process
  - n = k (ratio of specific heats) for an adiabatic or very fast process
  - n = 0.0 for an isobaric process
  - n = ∞ for an isovolumetric process
- Bulk Modulus for a gas is:

$$\beta = \rho_0 \frac{\partial P}{\partial \rho}\bigg|_{P_0, T_0} = \rho_0 \left[nC\rho^{n-1} \right]\Big|_{P_0, T_0} = \rho_0 n \frac{C\rho^n}{\rho}\bigg|_{P_0, T_0} = nP_0$$

- The bulk modulus of a gas is thus related to the absolute pressure of the gas.
- The bulk modulus of a liquid is on the order of 5000 to 15000 bar (1 bar = 1.0E5 N/m^2) compared to 1 to 10 bar for a gas.

Viscosity

- Absolute Viscosity (or dynamic viscosity) μ (dimensions Ft/L^2 or M/Lt) of a fluid represents the ability of the fluid to support a shear stress τ between a relative velocity *u* of the fluid and a solid boundary.

$$\mu = \frac{\tau}{\partial u / \partial y} = \frac{\text{shear stress}}{\text{shear rate}}$$

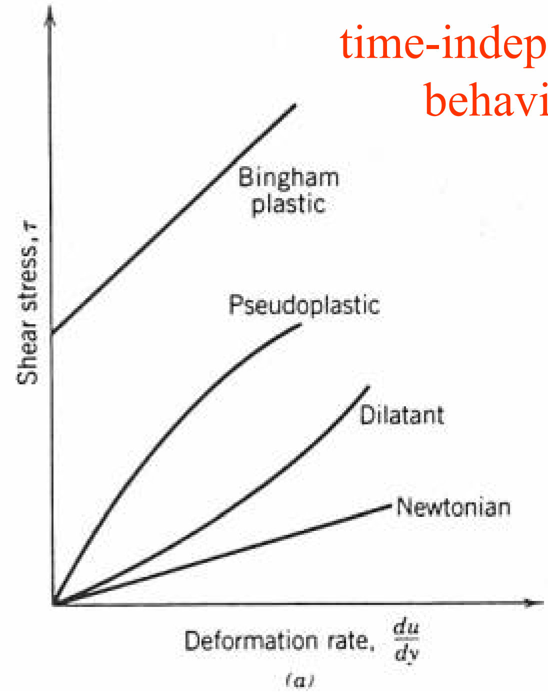
- A Newtonian fluid is a fluid in which shear stress is directly proportional to shear rate. This is not true for a non-Newtonian fluid.
- Kinematic Viscosity $\nu = \mu/\rho$ (dimensions L^2/t)

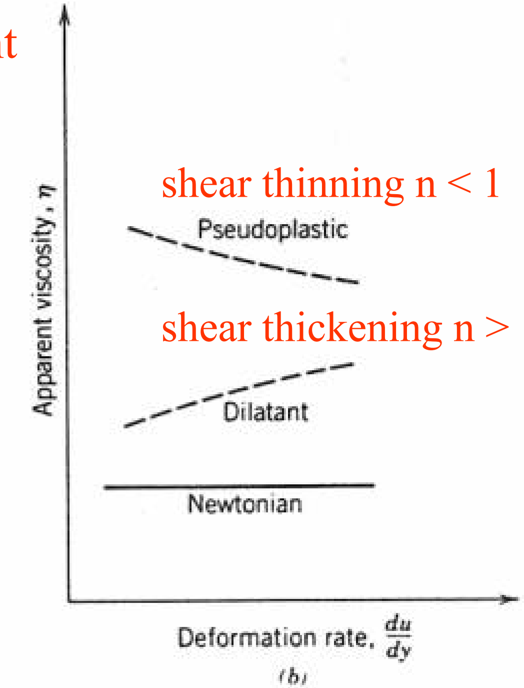
- Absolute viscosity of a liquid decreases markedly with temperature: $\mu = \mu_0 e^{-\lambda_L (T - T_0)}$
  - μ_0 and T_0 are values at the reference conditions.
  - λ_L is a constant that depends upon the liquid.
  - In a liquid, resistance to deformation is primarily controlled by cohesive forces among molecules.
- Absolute viscosity of a gas increases with temperature $\mu = \mu_0 + \lambda_G (T - T_0)$
  - μ_0 and T_0 are values at the reference conditions.
  - λ_G is a constant that depends upon the liquid.
  - In a gas, resistance to deformation is primarily due to the transfer of molecular momentum.

- The absolute viscosity of most gases is almost independent of pressure (from 1 to 30 bar).
- The viscosities of most liquids are not affected by moderate pressures, but large increases have been found at very high pressures.

Shear stress τ and apparent viscosity η as a function of deformation rate for one-dimensional flow of various non-Newtonian fluids

time-independent behavior





Power-Law Model:

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n = k\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy} = \eta\frac{du}{dy}$$

Bingham-Plastic Model:

$$\tau_{yx} = \tau_y + \mu_p\frac{du}{dy}$$

Other Categories:

time-dependent behavior $\begin{cases} \text{Thixotropic} \\ \text{Rheopectic} \end{cases}$

Viscoelastic

Under a constant applied shear stress:
Thixotropic – decrease in η with time
Rheopectic – increase in η with time

Propagation Speed

- Speed of propagation, c_0, of a pressure signal in a fluid depends on the bulk modulus and the density:

$$c_0 = \sqrt{\frac{\beta}{\rho}}$$

- The bulk modulus of a gas being perturbed at a high speed is the ratio of specific heats, k, times the absolute pressure of the gas, i.e., $\beta = kP$. Thus the ratio β/ρ reduces to

$$c_0 = \sqrt{kRT}$$

- The speed of sound in air at 25°C is 347 m/s and in a typical hydraulic fluid at 25°C is 1370 m/s.

Thermal Properties

- Specific heat of a fluid is the amount of heat required to raise the temperature of a unit mass of the fluid by 1 degree.
- The specific heat at constant pressure is C_p.
- The specific heat at constant volume is C_v.
- The specific heat ratio $k = C_p/C_v$.
- For some liquids, the specific heat ratio is approximately 1.04. For gases, k is larger, e.g., $k = 1.4$ for air.

Reynolds Number Effects

- Inertial flow forces $\propto \rho A v^2$

$$F = ma = \rho V a \propto \rho d^3 a$$

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} \quad \Rightarrow \quad a = v\frac{dv}{ds}$$

$$F \propto \frac{\rho d^3 v^2}{d} = \rho d^2 v^2 = \rho A v^2$$

- Viscous flow forces $\propto \mu d v$

$$\tau A = \mu \frac{du}{dy} A \propto \mu \frac{v}{d} d^2 = \mu v d$$

  - v is the velocity of the fluid
  - d is a characteristic dimension associated with the physical situation
- Flow conditions are often defined by means of a dimensionless ratio called the Reynolds Number, the ratio of inertial to viscous forces:

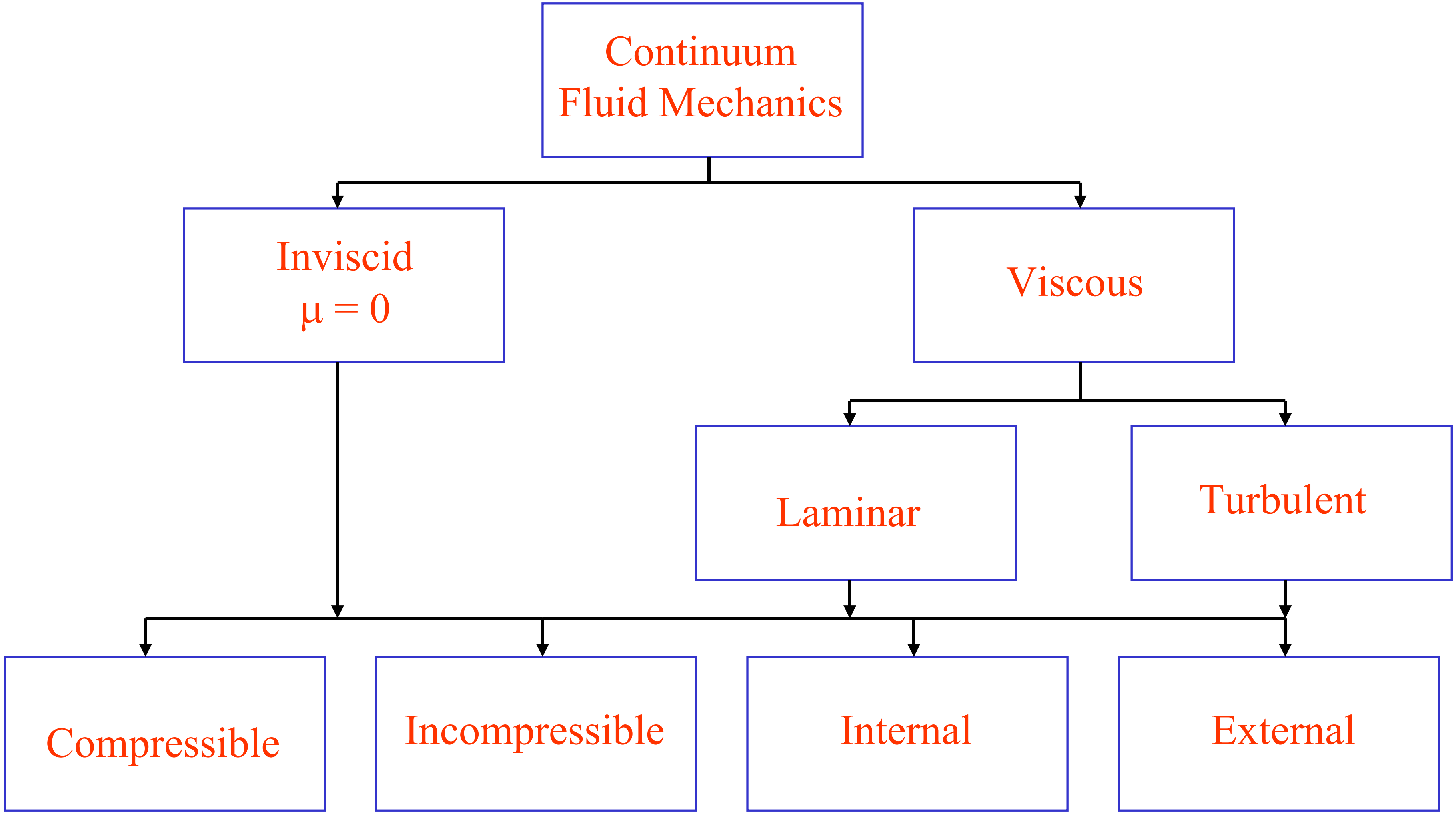
$$N_r = \frac{\rho d^2 v^2}{\mu d v} = \frac{vd}{\nu}$$

- Laminar Flow - smooth, streamlined flow where viscous forces dominate over inertial forces; no macroscopic mixing of adjacent fluid layers.
- Turbulent Flow - swirling flow where inertial forces dominate over viscous forces; macroscopic mixing of adjacent fluid layers.
- At small N_r (< 1400) flow is laminar.
- At high N_r (> 3000) flow is turbulent and has energy loss due to fluid collisions and mixing.
- For $1400 < N_r < 3000$, there is a transition from laminar to turbulent flow. Here flow depends on the local conditions and on the previous flow history.

- Laminar flow: pressure loss due to friction has a first-order relationship with the flow (analogous to electrical resistance in which voltage is linear with current).
- Turbulent flow: pressure loss becomes proportional to the square of the flow. There is no universal relationship between the stress field and the mean velocity field; one must rely heavily on semi-empirical theories and on experimental data.
- Systems with laminar flow are far simpler than those with turbulent flow; however, laminar flow is generally impractical for most systems because of the low pressures and small dimensions it requires.

Classification of Fluid Motions

Basis of observable physical characteristics of flow fields



- Incompressible Flow – density variations are negligible. Most liquid flows are essentially incompressible. However water hammer and cavitation are examples of the importance of compressibility effects in liquid flows. Gas flows may be considered incompressible provided the flow speeds are small relative to the speed of sound, e.g., < 30% of the speed of sound, which corresponds to 100 m/s for air at standard conditions.
- Compressible Flow – density variations are not negligible.

Basic Equations of Fluid Dynamics

- The basis laws governing fluid motion are:
  - Conservation of Mass
  - Newton's 2nd Law
  - Moment of Momentum
  - 1st Law of Thermodynamics
  - 2nd Law of Thermodynamics

Not all are required to solve any one problem.

- In addition one needs the equations of state or constitutive equations that describe the behavior of physical properties of fluids under given conditions.

- Methods of Analysis
  - A system or control volume must be defined before applying the basic equations in the analysis of a problem.
  - A *system* is defined as a fixed, identifiable quantity of mass; system boundaries separate the system from the surroundings and they may be fixed or movable, however, there is no mass transfer across the system boundaries. Heat and work may cross the system boundaries.
  - A *control volume* is an arbitrary volume in space through which fluid flows. The geometric boundary of the control volume is called the control surface, which may be real or imaginary, at rest or in motion, but it must be clearly defined before beginning analysis.

- The basic laws can be formulated in terms of infinitesimal systems or control volumes, in which case the resulting equations are *differential equations*, whose solutions provide detailed, point-by-point behavior of the flow.
- When we are interested in the gross behavior of a device, we use finite systems or control volumes. The *integral formulation* of the basic laws is more appropriate.
- The method of analysis depends on the problem!
- The method of description that follows a fluid particle is called the *Lagrangian* method of description.

- The field, or *Eulerian*, method of description focuses attention on the properties of the flow at a given point in space as a function of time. The properties of the flow field are described as functions of space coordinates and time. This method of description is a logical outgrowth of the fluid-as-a-continuum assumption.
- Two Basic Reasons for using the control volume formulation rather then the system formulation:
  - Since fluid media are capable of continuous deformation, it is often extremely difficult to identify and follow the same mass of fluid at all times, as must be done in applying the system formulation.
  - We are often interested not in the motion of a given mass of fluid, but rather in the effect of the overall fluid motion on some device or structure.

- <u>Conservation of Mass</u>
  - Mass of the system is constant.

$$\left(\frac{dM}{dt}\right)_{sys} = 0$$

$$M_{sys} = \int_{sys-mass} dm = \int_{sys-vol} \rho dV$$

- <u>Newton's 2nd Law</u>
  - For a system moving relative to an inertial reference frame, the sum of all external forces acting on the system is equal to the time rate of change of the linear momentum of the system.

$$\left(\vec{F}\right)_{sys} = \left(\frac{d\vec{P}}{dt}\right)_{sys} \qquad \overrightarrow{P_{sys}} = \int_{sys-mass} \vec{V} dm = \int_{sys-vol} \vec{V} \rho dV$$

- 1st Law of Thermodynamics: Conservation of Energy
  - Rate at which heat is added to the system plus the rate at which work is done on the system by the surroundings is equal to the rate of change of the total energy of the system:

$$\dot{Q} + \dot{W} = \left(\frac{dE}{dt} \right)_{sys}$$

$$E_{sys} = \int_{sys-mass} (e)\,dm = \int_{sys-vol} (e\rho)\,dV$$

$$e = u + \frac{v^2}{2} + gz$$

- Each system equation, written on a rate basis, involves the time derivative of an extensive property of the system. Let N = any arbitrary extensive property of the system. Let η = the corresponding intensive property (extensive property per unit mass).

$$N_{sys} = \int_{sys-mass} \eta \, dm = \int_{sys-vol} \eta \rho \, dV$$

- The equation relating the rate of change of the arbitrary extensive property, N, for a system to the time variations of this property associated with a control volume at the instant when the system and control volume coincide is:

$$\left(\frac{dN}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{CV} \eta \rho \, dV + \int_{CS} \eta \rho \vec{v} \bullet \overrightarrow{dA}$$

- This equation says: The time rate of change of the arbitrary extensive property of the system equals the time rate of change of the arbitrary extensive property within the control volume plus the net rate of efflux (rate of outflow minus the rate of inflow) of the arbitrary extensive property through the control surface.
- Note:
  - Velocity is measured relative to the control volume.
  - The time rate of change of the arbitrary extensive property within the control volume must be evaluated by an observer fixed in the control volume.

- Let's apply this equation to the three basic laws:
  - Conservation of Mass
  - Newton's 2nd Law
  - Conservation of Energy

- Conservation of Mass
  - The net rate of mass efflux through the control surface plus the rate of change of mass inside the control volume equals zero.

$$0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{v} \bullet \overrightarrow{dA}$$

  - Velocity is measured relative to the control volume.
  - $\rho \vec{v} \bullet \overrightarrow{dA}$ is a scalar: it is positive where flow is out through the control surface, negative where flow is in through the control surface, and zero where flow is tangent to the control surface.

- Newton's 2nd Law for an Inertial Control Volume
  - The sum of all forces acting on a non-accelerating control volume equals the rate of change of momentum inside the control volume plus the net rate of efflux of momentum through the control surface.

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{v}\rho dV + \int_{CS} \vec{v}\rho\vec{v}\bullet\overrightarrow{dA}$$

  - All velocities are measured relative to the control volume.
  - All time derivatives are measured relative to the control volume.

- 1st Law of Thermodynamics: Conservation of Energy

$$\dot{Q} + \dot{W} = \frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho\vec{v}\bullet\overrightarrow{dA}$$

$$e = u + \frac{v^2}{2} + gz$$

– $\dot{W}$ is positive when work is done on the control volume by the surroundings and this can take place only at the control surface.

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{normal} + \dot{W}_{shear} + \dot{W}_{other}$$

Lumped-Parameter Approach

- Lumped-Parameter Approach
  - Some areas have used this approach more than others, e.g., electrical systems are originally conceived as a combination of R, L, C, op-amps, and other integrated-circuit modules, while mechanical systems are rarely initially conceived in terms of a combination of K, B, and M, but rather as combination of mechanisms, actuators, sensors, and controllers.
  - Fluid and Thermal systems follow a similar pattern in that system dynamics may receive relatively light conscious emphasis during the early conceptual phases.

- In addition, due to the generally less-well-defined shapes of bodies of fluid and the fact that heat flow rarely is confined to such simple and obvious paths as current, these type systems may appear less well suited to the lumped-parameter viewpoint.
- System dynamics can preserve the identity of individual components while comprehending the entire system, and thus often gives insights into needed design changes at both the component and system levels.
- So, we consider system dynamics methods for fluid and thermal machines and processes, even though they initially seem less well suited to these more amorphous systems.
- There are many practical examples of actual hardware which have been successfully studied with this approach.

Passive Fluid Elements

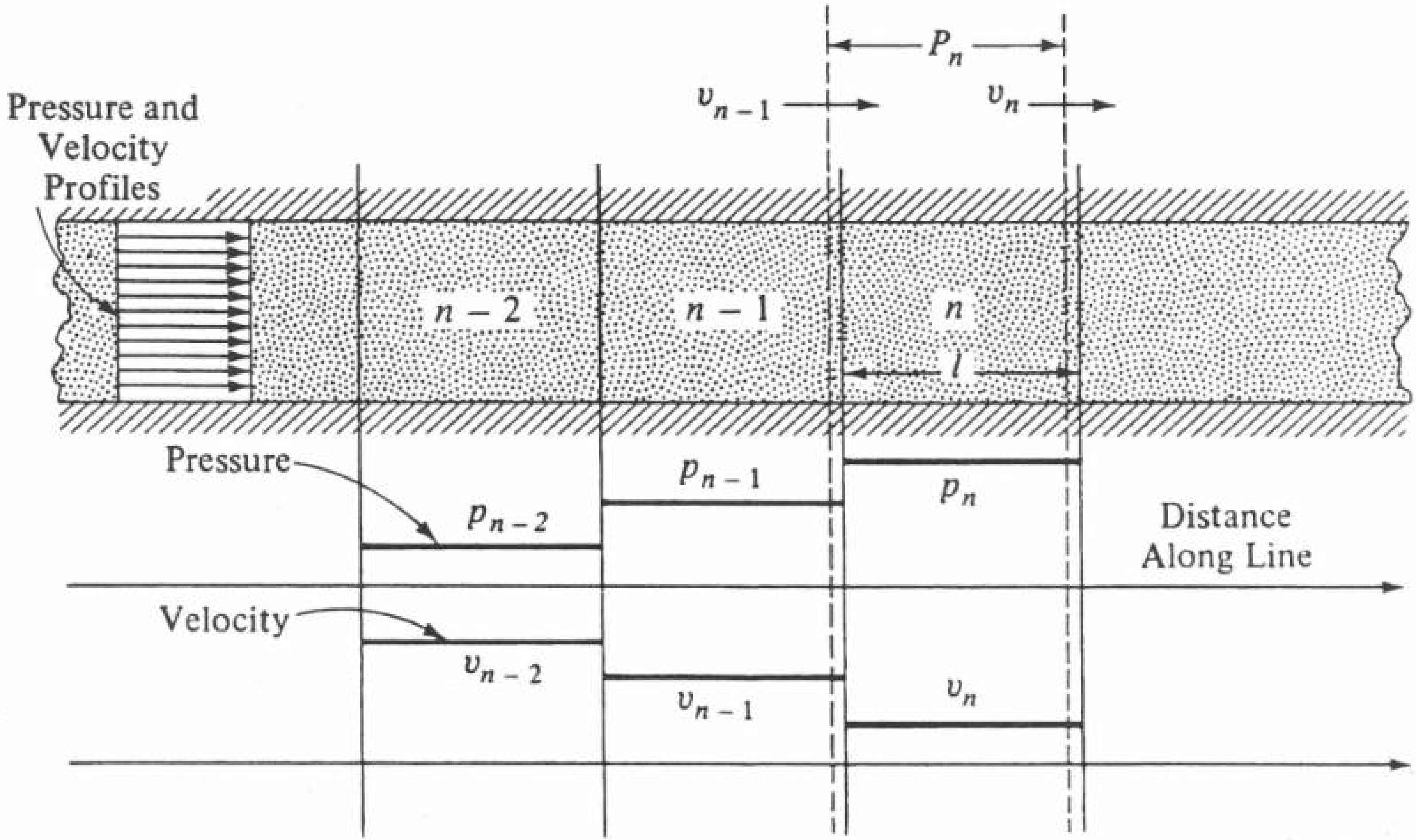
- Fluid Flow Resistance
- Fluid Compliance (Capacitance)
- Fluid Inertance (Inductance)

Fluid Flow Resistance

- Like mechanical friction and electrical resistance, this element performs the energy-dissipation function.
- The dissipation of fluid energy into heat occurs in all fluid devices to some extent.
- The simplest example is that of a fluid pipe.
  - Consider the flow of a fluid in a constant-area, rigid-walled conduit (as shown).
  - Variables of primary interest are: average fluid pressure p and the volume flow rate q.

– The average flow velocity *v* is defined as *q/A*, where *A* is the conduit cross-sectional area.
– The product of *p* and *q* is fluid power, just as mechanical power is *fv* and electrical power is *ei*.
– While the actual fluid pressure and velocity vary from point to point over the flow cross section in a real fluid flow, we assume a so-called one-dimensional flow model in which the velocity and pressure are uniform over the area.
– Thus, the average velocity and average pressure correspond numerically with the values at any point in the cross section. These averages are spacewise averages, not timewise averages.

Lumped Model of a Fluid Pipeline



- In a lumped-parameter dynamic analysis, the pipeline is broken into segments. Within each segment or lump, the pressure and velocity may vary arbitrarily with time, but are assumed uniform over the volume of the lump. By considering the behavior of one typical lump (the n^{th} lump) we are led to definitions of <u>all three basic fluid elements</u>.
- Experiments show that when a fluid is forced through a pipe at a constant flow rate, a pressure difference related to that flow rate must be exerted to maintain the flow. It is observed that it takes a larger pressure difference to cause a larger flow rate.
- In general, the relation between pressure drop and flow rate is nonlinear; however, some situations give a nearly linear effect.

– The pure and ideal fluid friction or fluid resistance element is defined as:

$$R_f \triangleq \frac{p_1 - p_2}{q} \left(\frac{\text{psi}}{\text{in}^3/\text{sec}} \right)$$

– For nonlinear fluid resistances, we can define linearized values in the neighborhood of an operating point.

– Now consider Conservation of Mass applied to the nth lump over an infinitesimal time interval *dt*. During the time interval *dt*, the difference between mass flow into and out of the lump must equal the additional mass stored in the lump. Mass enters the lump from the left at a rate $Av_{n-1}\rho$ and leaves at the right at a rate $Av_n\rho$, where ρ is the fluid mass density, which we treat as being constant, corresponding to a constant operating-point pressure and temperature.

– For constant density, conservation of mass is the same as conservation of volume:

$$\left(Av_{n-1} - Av_n\right)dt = dV = \frac{V}{\beta}dp_n = \frac{A\ell}{\beta}dp_n$$

$$\left(q_{n-1} - q_n\right)dt = \frac{A\ell}{\beta}dp_n$$

$$p_n = \frac{1}{C_f}\int\left(q_{n-1} - q_n\right)dt$$

$$C_f \triangleq \frac{A\ell}{\beta} \triangleq \text{fluid compliance (capacitance)}$$

– We can easily see the electric-fluid analogy: electric current analogous to net volume flow rate, pressure analogous to voltage drop, and electrical capacitance analogous to fluid capacitance.

– Newton's 2nd Law states that the force (pressure times area) difference between the left and right ends of a lump must equal the lump mass times its acceleration. For the nth lump the result is:

$$Ap_{n-1} - Ap_n - AR_f q_n = \rho A \ell \frac{dv_n}{dt} = \rho \ell \frac{dq_n}{dt}$$

$$\left(p_{n-1} - p_n\right) - R_f q_n = \frac{\rho \ell}{A} \frac{dq_n}{dt}$$

– Since this equation contains both resistance (friction) and inertance (inertia) effects, we consider each (in turn) negligible, to separate them. With zero fluid density (no mass) we have:

$$\left(p_{n-1} - p_n\right) = R_f q_n$$

– If the resistance (friction) were zero, we have:

$$\left(p_n - p_{n-1}\right) = \frac{\rho \ell}{A} \frac{dq_n}{dt} = I_f \frac{dq_n}{dt}$$

$$I_f \triangleq \frac{\rho \ell}{A} \triangleq \text{fluid inertance}$$

– Again we see the electric-fluid analogy: pressure drop analogous to voltage drop, volume flow rate analogous to current, and fluid inertance analogous to electric inductance.

– We will return to the fluid compliance and inertance elements in more detail, since they occur in more general contexts, not just pipelines. They were introduced here to illustrate that the three elements are always present whenever a body of fluid exists.

- Whether all three will be included in a specific system model depends on the application and the judgment of the modeler.
- Let's consider fluid resistance in a more general way. When a one-dimensional fluid flow is steady (velocity and pressure at any point not changing with time), the inertance and compliance cannot manifest themselves (even though they exist), and only the resistance effect remains. We can thus experimentally determine fluid resistance by steady-flow measurements of volume flow rate and pressure drop, or if we attempt to calculate fluid resistance from theory, we must analyze a steady-flow situation and find the relation between pressure drop and volume flow rate. If a nonlinear resistance operates near a steady flow q_0, we can define a linearized resistance, good for small flow and pressure excursions from that operating point.

- Just as in electrical resistors, a fluid resistor dissipates into heat all the fluid power supplied to it. Fluid power at a flow cross section is the rate at which work is done by the pressure force at that cross section:

$$\text{Power} = pAv = pq$$

- For our assumed one-dimensional incompressible flow (volume flow rate same at both sections), the power dissipated into heat is:

$$q\left(p_1 - p_2\right) = q\Delta p = q^2 R_f = \frac{\Delta p^2}{R_f}$$

- While we can often determine flow resistance values by experimental steady-flow calibration, it is desirable to be able to calculate from theory, before a device has been built, what its resistance will be. For certain simple configurations and flow conditions, this can be done with fairly good accuracy.

- Refer to the previous discussion on Laminar and Turbulent flow and the Reynolds Number.
- Laminar flow conditions produce the most nearly linear flow resistances and these can be calculated from theory, for passages of simple geometrical shape.
- For example, a long, thin flow passage called a capillary tube of circular cross section has a fluid resistance:

$$q = \frac{\pi D^4}{128 \mu L} \Delta p$$

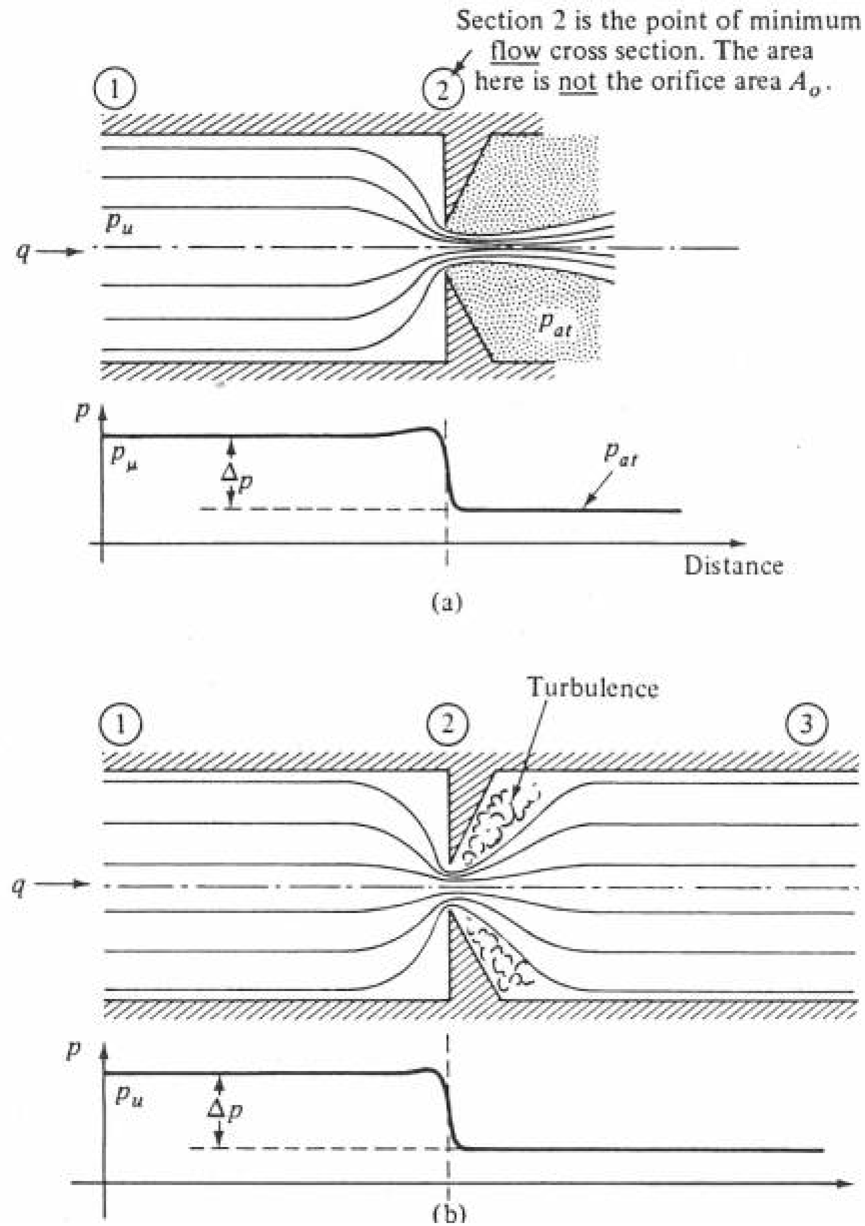
$$R_f \triangleq \frac{\Delta p}{q} = \frac{128 \mu L}{\pi D^4}$$

- This is valid for laminar flow with end effects neglected.

- Note that in the above calculations and experimental measurements of flow resistances, the approach has been to use formulas relating flow rate and pressure for steady flows as if they held for general (unsteady) flows. This approach is widely used, and usually of sufficient accuracy; however, it should be recognized as an approximation.
- Now consider orifices, where resistance is concentrated in a short distance. The pressure drop across an orifice is basically due to a conversion from the form of fluid power, pressure times volume flow rate, to the power of kinetic energy. From conservation of energy, for a level flow of a frictionless incompressible fluid:

pressure/flow power + kinetic energy power = constant

Characteristics of Orifice Flow



– Considering any two locations 1 and 2, we may write:

$$p_1 q + (\text{KE per unit time})_1 = p_2 q + (\text{KE per unit time})_2$$

$$p_1 q + \frac{\frac{\rho A L v_1^2}{2}}{\frac{AL}{q}} = p_2 q + \frac{\frac{\rho A L v_2^2}{2}}{\frac{AL}{q}} \quad \Rightarrow \quad p_1 - p_2 = \Delta p = \frac{\rho}{2}\left(v_2^2 - v_1^2\right)$$

– Since $q = A_1 v_1 = A_2 v_2$,

$$\frac{2\Delta p}{\rho} = \left(\frac{q}{A_2}\right)^2 - \left(\frac{q}{A_1}\right)^2 = \frac{1-\left(A_2/A_1\right)^2}{A_2^2} q^2$$

$$q = \frac{A_2}{\sqrt{1-\left(A_2/A_1\right)^2}} \sqrt{\frac{2\Delta p}{\rho}}$$

Basic pressure/flow relation (nonlinear) for an orifice

- Because of turbulence in the flow, viscosity of real fluids, and difficulties in measuring the areas of the fluid stream, the practical formula used to predict the pressure/flow relation for orifices in pipelines uses experimental data, defines the pressure drop as that between stations 1 and 3, and uses only the cross-sectional area of the hole in the metal orifice:

$$q = C_d A_o \sqrt{\frac{2\Delta p}{\rho}}$$

- The value of the orifice discharge coefficient C_d depends mainly on the Reynolds number and the area ratio $A_{pipe} / A_{orifice}$.

– The fluid resistances discussed so far are all intended to be essentially constant in numerical form. Many fluid systems require adjustable resistances, and often these take the form of some kind of valve used to smoothly modulate the flow rate. The vast majority of valves have a square-root type of pressure/flow relation and usually require experimental calibration if we want a reasonably accurate flow model. The flow modulation is achieved by somehow varying the “orifice” area A_o. A complete calibration would give a family of flow rate versus pressure drop curves, one for each flow area.

Fluid Compliance

- We have already seen that a fluid itself, whether a liquid or gas, exhibits compliance due to its compressibility.
- Certain devices may also introduce compliance into a fluid system, even if the fluid were absolutely incompressible.
  - Rubber hoses will expand when fluid pressure increases, allowing an increase in volume of liquid stored.
  - Accumulators use spring-loaded cylinders or rubber air bags to provide intentionally large amounts of compliance.

- A simple open tank exhibits compliance, since an increase in volume of contained liquid results in a pressure increase due to gravity.

- In general, the compliance of a device is found by forcing into it a quantity of fluid and noting the corresponding rise in pressure. For liquids, the input quantity is a volume of fluid V, and the ideal compliance is:

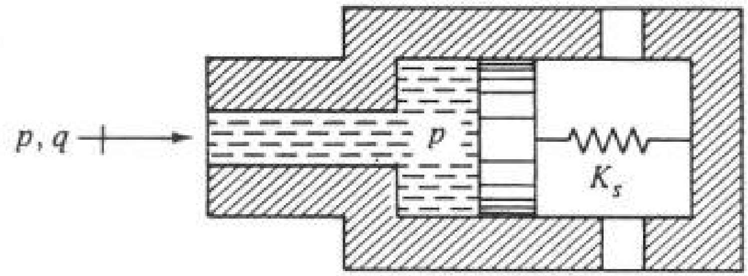
$$\text{Fluid Compliance} \triangleq C_f \triangleq \frac{V}{p} \triangleq \frac{\int q\,dt}{p}$$

  - For nonlinear compliances, the actual p-V curve can be implemented in a computer lookup table, or the local slope can be used to define an incremental compliance.

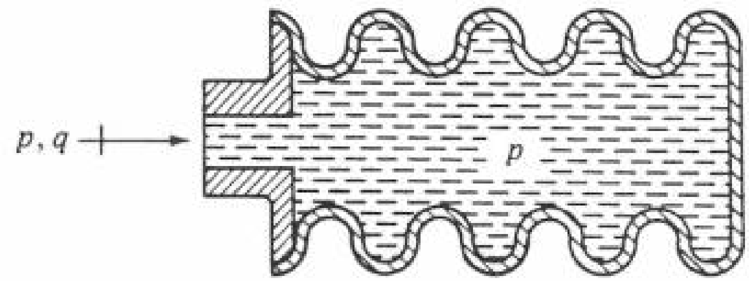
- We previously calculated the compliance of a section of hydraulic line, due to bulk modulus of the liquid itself, as: $\dfrac{A\ell}{\beta}$
- Additional contributions to compliance which may be significant here are due to entrained air bubbles and the flexibility of the tubing.
- Accumulators are devices intentionally designed to exhibit fluid compliance, e.g., spring-loaded piston and cylinder; flexible metal bellows; nitrogen-filled rubber bag. Some devices can store large amounts of fluid energy and are widely used for short-term power supplies, pulsation smoothing, and to reduce pump size in systems with intermittent flow requirements.

Accumulators

(a) spring-loaded piston and cylinder
(b) flexible metal bellows
(c) nitrogen-filled rubber bag



(a)



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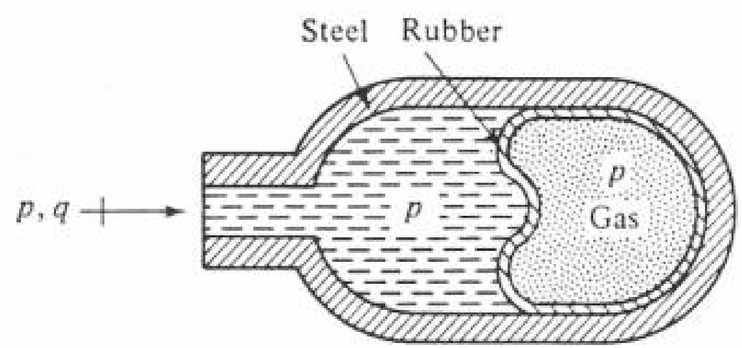
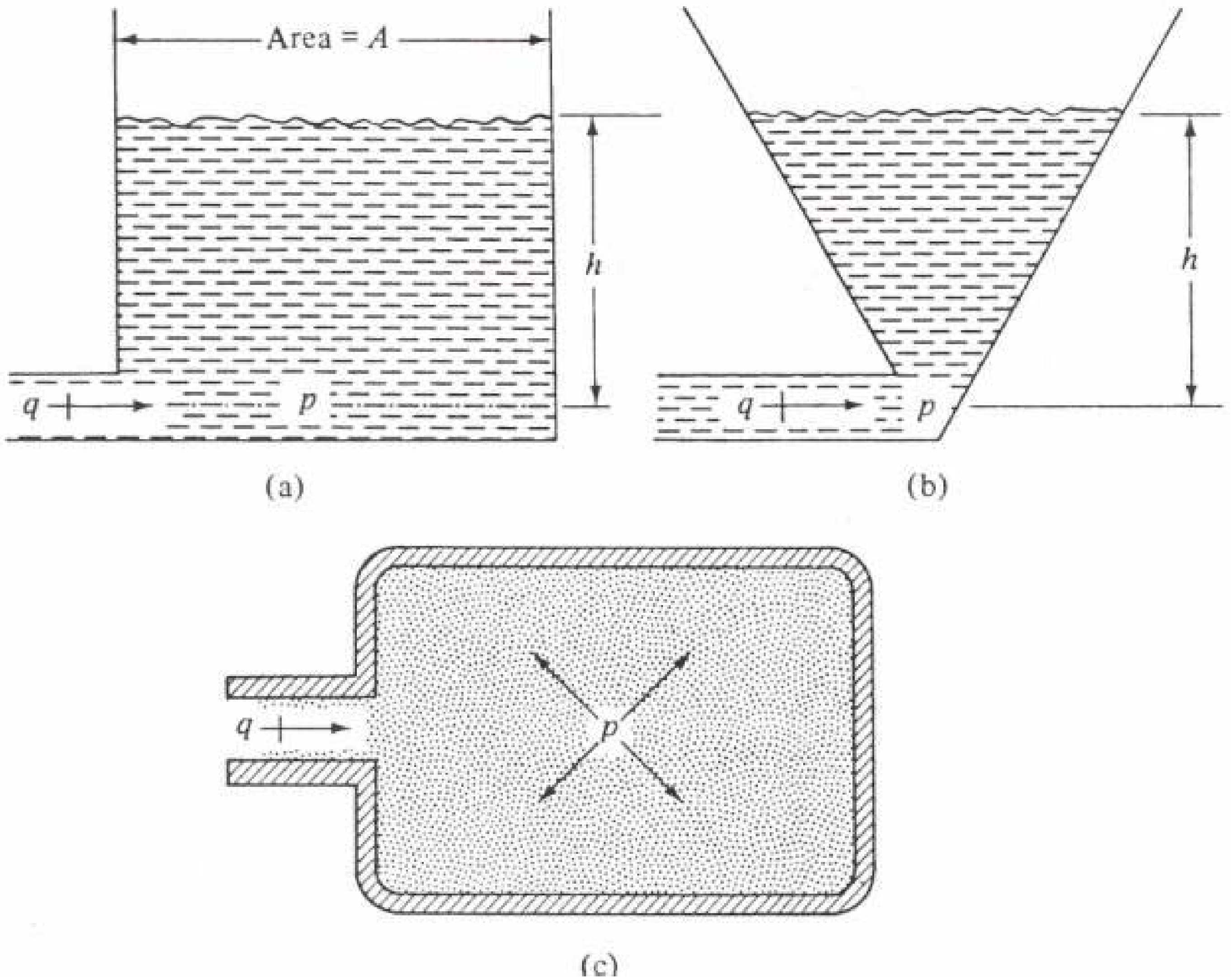


(c)

- Consider a vertical cylindrical tank of cross-sectional area A supplied with a volume flow rate q; the pressure at the tank inlet is p, liquid height is h. The vertical motion of the liquid in such tanks is usually slow enough that the velocity and acceleration have negligible effects on the pressure p and it is simply given by $p = \gamma h$, where γ is the specific weight of the liquid.
- Think of the tank as a large-diameter vertical pipe and R_f (velocity effect) and I_f (acceleration effect) are negligible relative to the height effect.
- If we add a volume V of liquid to the tank, the level h goes up an amount V/A and the pressure rises an amount $V\gamma/A$.

Liquid and Gas Tanks as Fluid Compliances



– The tank compliance is thus:

$$C_f = \frac{\text{volume change}}{\text{pressure change}} = \frac{V}{V\gamma / A} = \frac{A}{\gamma}\left(\frac{\text{in}^3}{\text{psi}}\right)$$

– For non-cylindrical tanks the compliance effect is nonlinear, but can be linearized in the usual way if desired.

– Consider a rigid tank of volume V containing a gas at pressure p. For slow (isothermal) pressure changes in which fluid density is nearly constant, we may write $pV = MRT$. If we force a mass $dM = \rho dV$ of gas into the tank we cause a pressure change dp given by:

$$dp = \frac{RT}{V} dM = \frac{RT}{V} \rho dV = \frac{RT}{V} dV \frac{p}{RT}$$

$$C_f \triangleq \left[\frac{dV}{dp} \right]_{p=p_0} = \frac{V}{p_0} \left(\frac{in^3}{psi} \right)$$

– This is a linearized compliance useful for small changes near an operating point p_0. For rapid (adiabatic) but still small pressure changes, analysis shows the compliance is:

$$C_f = \frac{V}{kp_0} \quad \text{where k is the ratio of specific heats}$$

- Fluid Capacitance relates how fluid energy can be stored by virtue of pressure.
- Law of Conservation of Mass (Continuity Equation) for a control volume:

$$\dot{m}_{net} = \frac{d}{dt}\left(M_{cv}\right) = \frac{d}{dt}\left(\rho_{cv} V_{cv}\right)$$

$$\dot{m}_{net} = \rho Q_{net} = \rho_{cv}\dot{V}_{cv} + V_{cv}\dot{\rho}_{cv}$$

- If all densities of the system are equal to ρ, then:

$$Q_{net} = \dot{V} + \frac{V}{\rho}\dot{\rho}$$

- This assumption is justified for incompressible fluids and is quite accurate for compressible fluids if pressure variations are not too large and the temperature of flow into the control volume is almost equal to the temperature of flow out of the control volume.
- Now $\rho = \rho(P, T) \Rightarrow \dot{\rho} = \frac{\partial \rho}{\partial P}\frac{\partial P}{\partial t} = \frac{\rho_0}{\beta}\dot{P}$ $\qquad \beta = \rho_0 \left.\frac{\partial P}{\partial \rho}\right|_{P_0, T_0}$

$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv}$$

- If the container is rigid, then $\dot{V} = 0 \Rightarrow Q = \frac{V}{\beta}\dot{P}_{cv} = C_f \dot{P}_{cv}$
- C_f is the fluid capacitance. Any large volume of a compressible fluid becomes a capacitance.

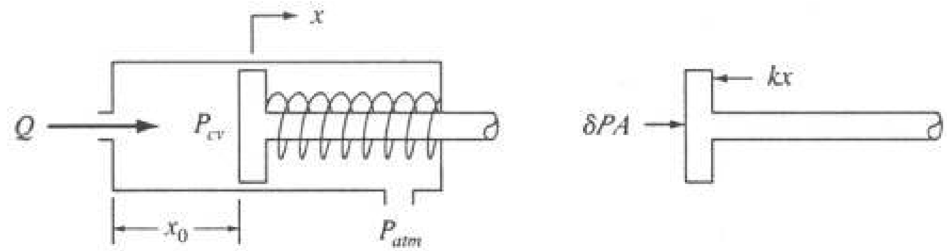
- The continuity equation gives the differential equation for the pressure inside the control volume. The following equation should be used in systems analysis (for systems with an inlet flow with the same density as the control volume), without repeating its derivation:

$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv}$$

$$\dot{P}_{CV} = \frac{\beta}{V}\left(Q - \dot{V}\right)$$

- The term $\dot{V}$ can take several forms depending upon the exact configuration of the system (control volume).

- For Example:



  - A spring-loaded piston of stiffness k and Area A can be used as a capacitor.
  - By applying a force balance equation, neglecting inertial and frictional effects, we obtain:

$$A\left(P_{cv} - P_{atm}\right) - kx = A\delta P - kx = 0$$

$$\dot{x} = \frac{A}{k}\delta\dot{P}$$

  - The volume of the cylinder is:

$$V = A\left(x + x_0\right)$$

$$\dot{V} = A\dot{x}$$

  - x_0 is the minimum stroke representing the dead volume space in the actuator and fittings.

– Substitution yields:

$$\dot{V} = \frac{A^2}{k}\delta\dot{P}$$

– The continuity equation can now be written as:

$$Q = \dot{V} + \frac{V}{\beta}\dot{P}_{cv}$$

$$Q = \left(\frac{V}{\beta} + \frac{A^2}{k}\right)\delta\dot{P} = C_f\delta\dot{P}$$

– The total capacitance is a combination of the fluid compressibility effects and the mechanical compliance of the container.

– Thus, fluid capacitance can be either a compliant container or a volume of fluid.

- Increasing A or decreasing k increases the mechanical capacitance term, and increasing V or decreasing β increases the fluid compressibility capacitance.
- Capacitors for liquids are called accumulators. Typical accumulators are spring-loaded pistons, bellows, and gas-filled bladders for hydraulic systems.
- Since β is large for incompressible fluids, mechanical types of capacitors are used. To obtain significant compressibility capacitance with a liquid, V would have to be very large.
- Capacitance of gaseous systems can be that of mechanical capacitors or volume-type capacitance, since β is low for compressible fluids.

- The effect of fluid capacitance must be considered relative to the rest of the system. For example, the resistance connected to the capacitor and the bandwidth of interest determine how significant the capacitive effects are.

Inertance (Fluid Inductance)

- While devices for introducing resistances (orifices, capillaries) and compliance (tanks, accumulators) are often intentionally designed into fluid systems, the inertia effect is more often than not a parasitic one.
- The inertance of pipes is perhaps most commonly encountered and we will emphasize it.
  - Any flowing fluid has stored kinetic energy because of its density (mass) and velocity. The inertance of a finite-size lump of fluid represents a summing up of this kinetic energy over the volume of the lump.

– The simplest assumption possible here is that of one-dimensional flow where all the fluid particles have identical velocities at any instant of time. Since every fluid particle has the same velocity, a lump of fluid can be treated as if it were a rigid body of mass $M = \rho AL$.

– A pressure drop Δp across a pure inertance element will cause a fluid acceleration according to Newton's Law:

$$A\Delta p = \rho AL\frac{dv}{dt} = \rho AL\left(\frac{1}{A}\frac{dq}{dt}\right)$$

$$\Delta p = \frac{\rho L}{A}\frac{dq}{dt} \triangleq I_f\frac{dq}{dt}$$

$$I_f \triangleq \frac{\rho L}{A}$$

- This is analogous to $e = L(di/dt)$ for inductance in electrical systems and $f = M(dv/dt)$ for mass in mechanical systems.
- This equation is valid for liquids and gases; however, if a gas is used, the density must be evaluated at the upstream conditions.
- The significance of fluid inductance must also be evaluated relative to the rest of the system and the bandwidth or frequency of interest.

Lumped vs. Distributed Fluid System Models

- Lumped-parameter models are always approximations to the more-correct distributed models.
- The comparison of different models for a system is often best done in terms of frequency response.
- Any practical dynamic system will experience input signals whose maximum frequency will be limited to a definite value.
- In lumped modeling, the variation of pressure and velocity along the length of a pipeline was assumed a stepwise one. Within a given lump there was no variation, but pressure and velocity did change when we went to a neighboring lump.

- All distributed models allow a smooth, not stepwise, variation, which of course is more correct.
- It is clear that as a lumped model uses more and smaller lumps, the stepwise variation more nearly approximates the true smooth variation.
- How many lumps are needed to get accurate results with a lumped model?
- Experience with many kinds of systems shows that if we choose 10 lumps per wavelength of the highest frequency, we usually get good results. That is, a stepwise variation is an acceptable approximation to a sine wave if there are 10 steps per wavelength. If we decide that our lumped model needs to be good for excitations of higher frequency than we originally planned, the lumps must get smaller and there must be more of them.

- Once we have a formula for the velocity of propagation (the speed with which a disturbance propagates through the medium) and choose the highest frequency of interest, we can pick a size and number of lumps which will give good accuracy up to that frequency, using the "10 lumps per wavelength" guideline, where the wavelength λ is the velocity of propagation c divided by the frequency f. Higher operating frequencies require more and smaller lumps.
- Remember why we use lumped-parameter models. They can be solved easily for the time response to any form of input and they also allow easy simulation with standard software.

Fluid Impedance

- Most fluid systems do not really require the separation of pressure/flow relations into their resistive, compliant, and inertial components; this separation is mainly one of analytical convenience.
- For complex fluid systems where experimental measurements may be a necessity, the measurement of overall pressure/flow characteristics has become a useful tool.
- The term fluid impedance is directly analogous to mechanical and electrical impedance and is defined as the transfer function relating pressure drop (or pressure), as output, to flow rate, as input.

- **Fluid Resistance**

$$\frac{\Delta p}{q}(D) = R_f$$

$$\frac{\Delta p}{q}(i\omega) = R_f$$

- **Fluid Compliance**

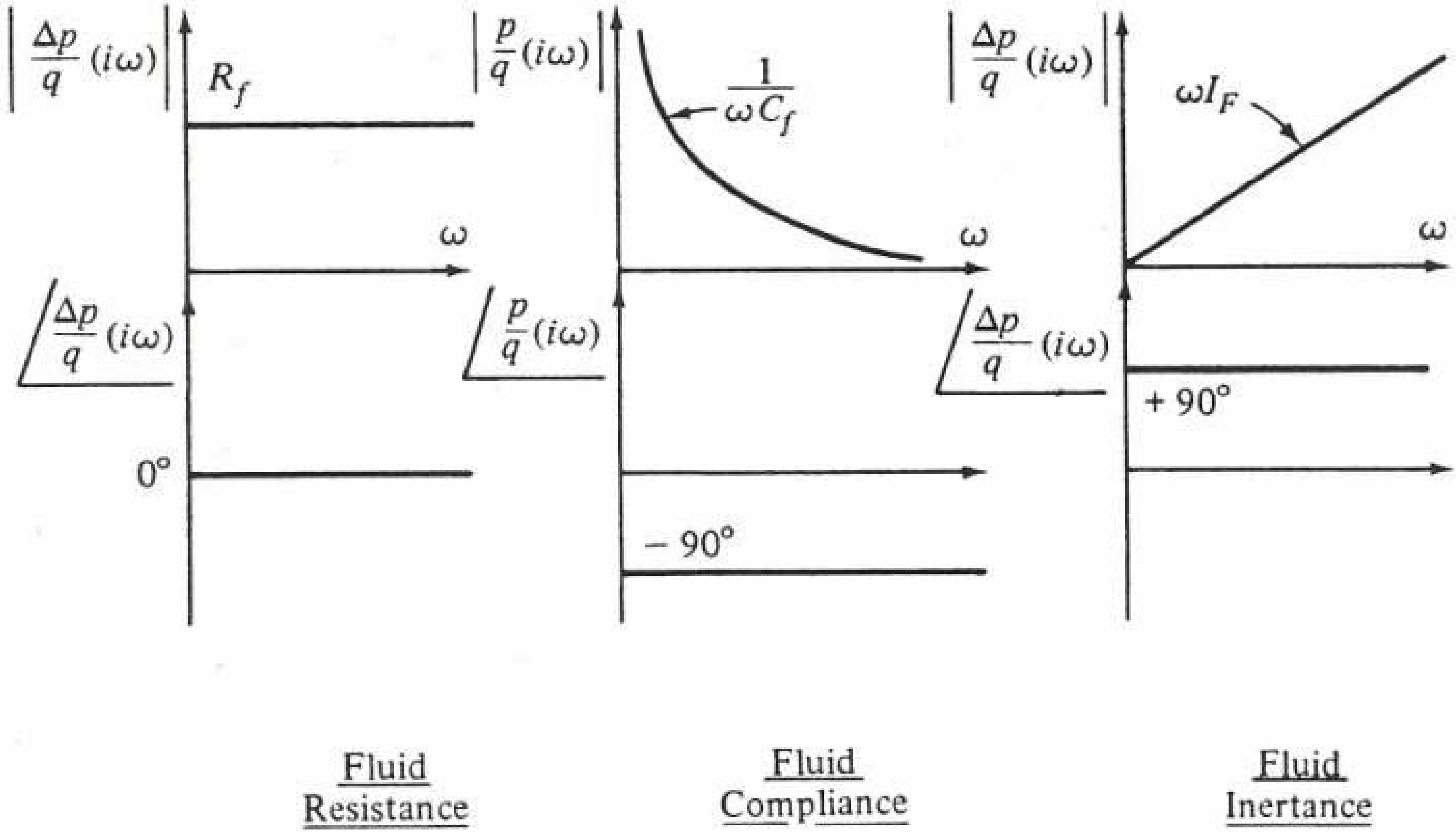
$$\frac{\Delta p}{q}(D) = \frac{1}{C_f D}$$

$$\frac{\Delta p}{q}(i\omega) = \frac{1}{\omega C_f}\angle -90^\circ$$

- **Fluid Inertance**

$$\frac{\Delta p}{q}(D) = I_f D$$

$$\frac{\Delta p}{q}(i\omega) = \omega I_f \angle +90^\circ$$



Fluid Impedance of the Three Basic Elements

- If the fluid impedance is known as an operational transfer function, one can calculate the response to any given input by solving the corresponding differential equation.
- If the fluid impedance is measured by the frequency response technique, we then do not have a transfer function in equation form, we only have curves. The response to sinusoidal inputs is of course easily calculated from such curves.
- Suppose, however, that we want to find the response to an input which is not a sine wave but rather has a time variation of arbitrary shape. Two methods are available:
  - Curve-fit the measured frequency-response curves with analytical functions, trying different forms and numerical values until an acceptable fit is achieved. Software to expedite such curve-fitting is available. Having a formula for the transfer function is the same as having the system differential equation.

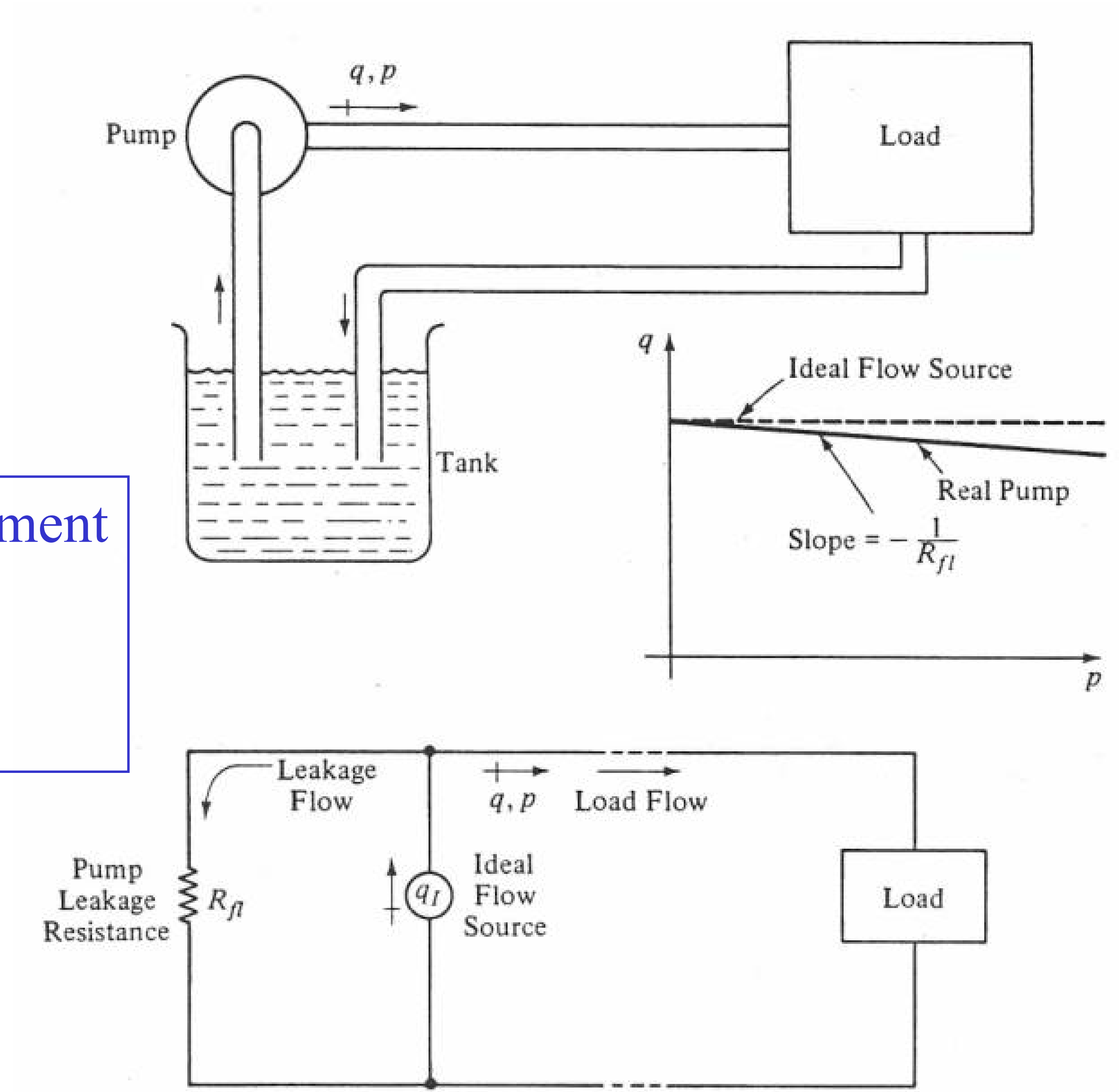
- Use the measured frequency-response curves directly, without any curve fitting. One must compute the Fourier transform of the desired time-varying input signal to get its representation in the frequency domain. This operation converts the time function into its corresponding frequency function, which will be a complex number which varies with frequency. This complex number is multiplied, one frequency at a time, with the complex number, whose magnitude and phase can be graphed versus frequency. This new complex number is the frequency representation of the output of the system. The final step is the inverse Fourier transform to convert this function back into the time domain, to give the system output as a specific, plottable, function of time.

- This discussion applies to any linear, time-invariant, dynamic system, not just fluid systems. That is, if we can measure the frequency response, we can get the response to any form of input, not just sine waves.

Fluid Sources: Pressure and Flow Rate

- An *ideal pressure source* produces a specified pressure at some point in a fluid system, no matter what flow rate might be required to maintain this pressure.
- An *ideal flow source* produces a specified flow rate, irrespective of the pressure required to produce this flow.
- In fluid systems, the most common source of fluid power is a pump or compressor of some sort.
- A positive-displacement liquid pump draws in, and then expels, a fixed amount of liquid for each revolution of the pump shaft. When driven at a constant speed, such a pump closely approximates an ideal constant-flow source over a considerable pressure range.

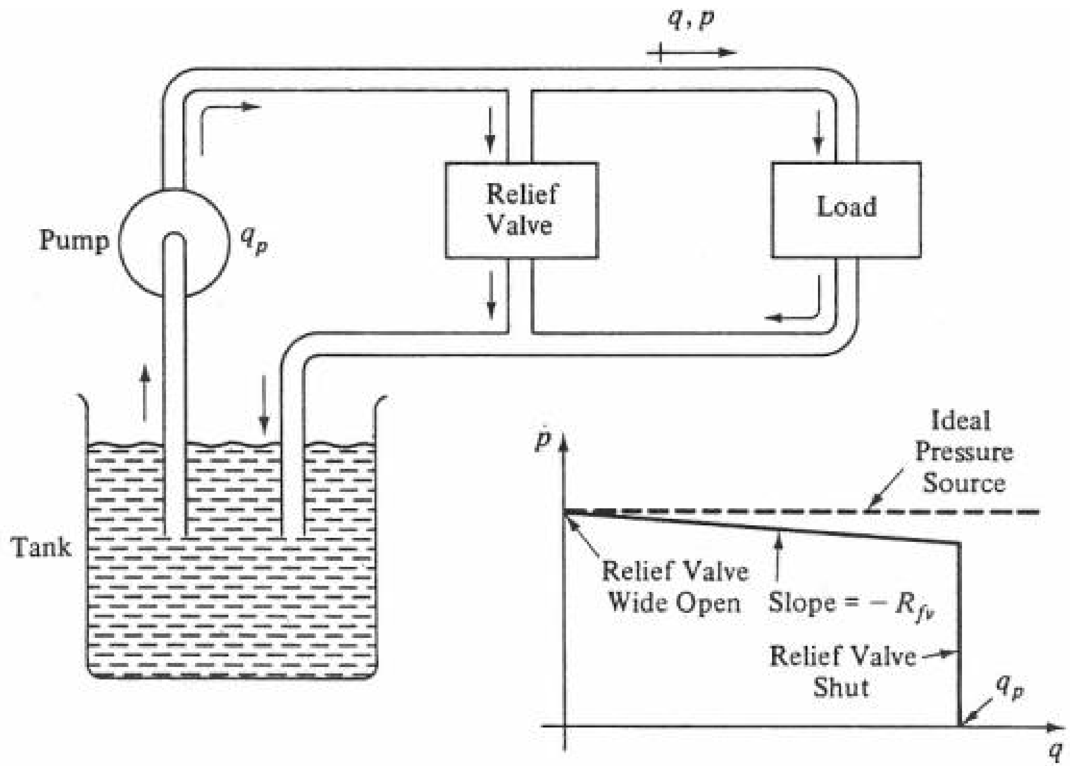
- Its main departure from ideal behavior is a decrease in flow rate, due to leakage through clearance spaces, as load pressure increases. This leakage flow is proportional to pressure; thus one can represent a real pump as a parallel combination of an ideal flow source and a linear (and large) flow resistance R_{fl}.
- If the inlet flow impedance of the load is low relative to R_{fl}, most of the flow goes into the load rather than the pump leakage path, and the pump acts nearly as an ideal flow source.



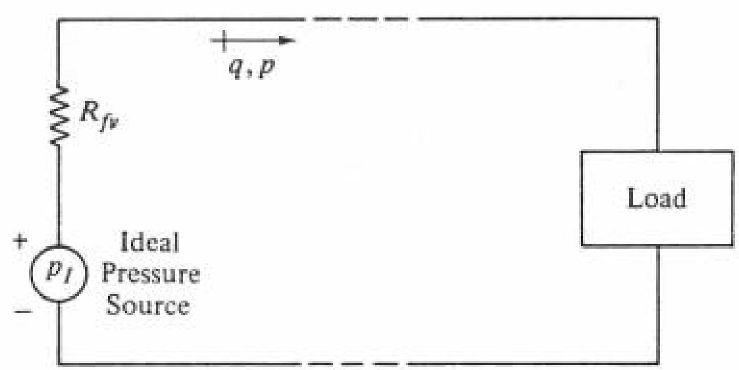
Positive-Displacement
Pump
as a
Flow Source

- When we need to manipulate a flow rate as function of time, several possibilities exist.
  - A fixed-displacement pump may be driven at a time-varying speed. An electric motor drives the pump shaft, a flow sensor measures the flow, and a feedback controller compares the desired flow with the measured flow and commands the motor to change speed so as to keep the actual flow close to the desired at all times.
  - Instead of a fixed-displacement pump, a variable-displacement pump could be used. Here the pump shaft speed is constant, but pump output per revolution can be varied, while the pump is running. A stroking mechanism allows flow rate to be varied smoothly and quickly from full flow in one direction, through zero flow, to full flow in the reverse direction. The stroking mechanism could be driven directly or we could again use a feedback scheme.

- By combining a positive-displacement pump with a relief valve, one can achieve a practical constant-pressure source. This real source will not have the perfect characteristic of an ideal pressure source, but can be modeled as a combination of an ideal source with a flow resistance.
- A relief valve is a spring-loaded valve which remains shut until the set pressure is reached. At this point it opens partially, adjusting its opening so that the pump flow splits between the demand of the load and the necessary return flow to the tank. To achieve this partial opening against the spring, the pressure must change slightly; thus we do not get an exactly constant pressure.
- This real source can be modeled as a series combination of an ideal pressure source with a small flow resistance.



q, p
Pump
q_p
Relief Valve
Load
Tank
p
Ideal Pressure Source
Relief Valve Wide Open
Slope = $-R_{fv}$
Relief Valve Shut
q_p
q



q, p
R_{fv}
+
p_I
−
Ideal Pressure Source
Load

- These examples do not exhaust the possibilities with regard to power sources in fluid systems, but they should give some idea of how real sources may be modeled in terms of ideal sources and passive elements.
- Other fluid power sources encountered in practice include centrifugal pumps, accumulators (used for short-term power supplies), elevated tanks or reservoirs (gravity is the energy source), etc.

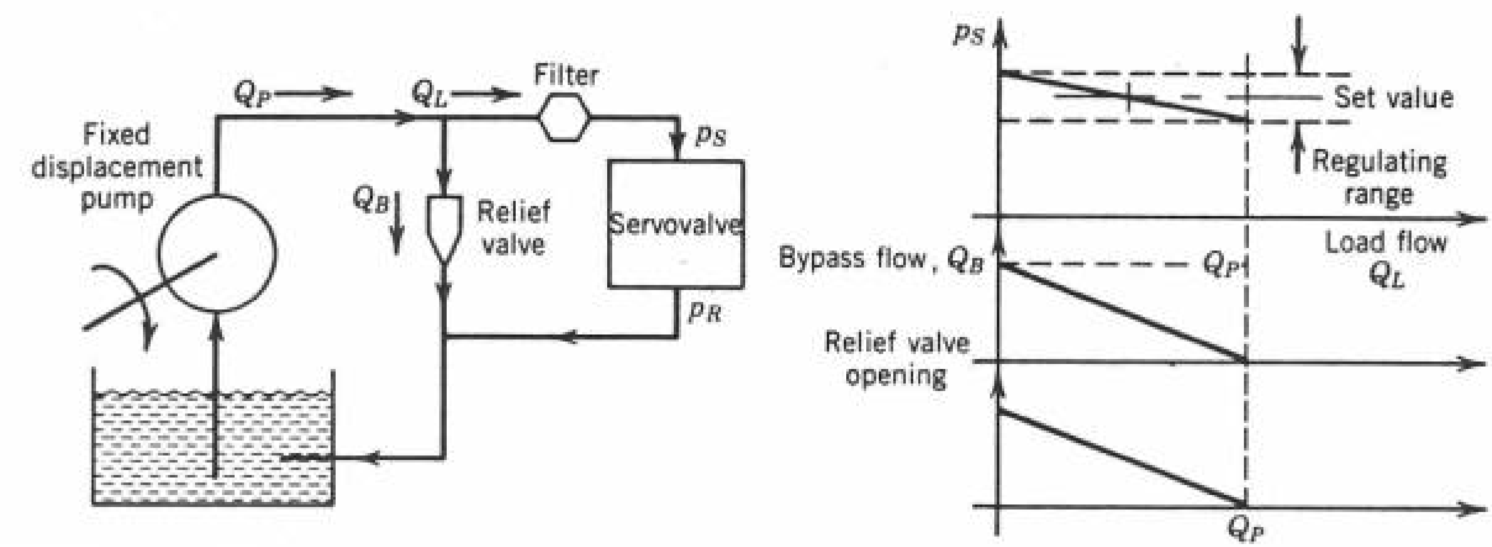
Mechatronics Exercise:
Modeling, Analysis, & Control of an Electrohydraulic Valve-Controlled Servomechanism

Introduction

- Although a wide variety of detailed hydraulic control schemes are in use, a useful classification is that of *pump-controlled* versus *valve-controlled* systems.
  - *Pump-controlled systems* are usually relatively high power (above 10 or 20 hp) applications, where efficiency is economically significant and response speed requirements are modest (less than 10 Hz frequency response).

- *Valve-controlled systems* are faster but are generally quite inefficient. For a low-power system, inefficiency has little economic impact. For fast high-power systems where speed specifications can only be met by valve control, the economic cost of low efficiency must be accepted.
- In *pump-controlled systems*, the fluid power supply must be included in the system model, while the analysis of *valve-controlled systems* can proceed without consideration of power supply details if one assumes the existence of a power source of constant supply pressure, p_s, irrespective of flow demand. Power supplies that approximate this behavior are available in several different forms that trade off complexity, cost, efficiency, and static/dynamic pressure regulation accuracy.

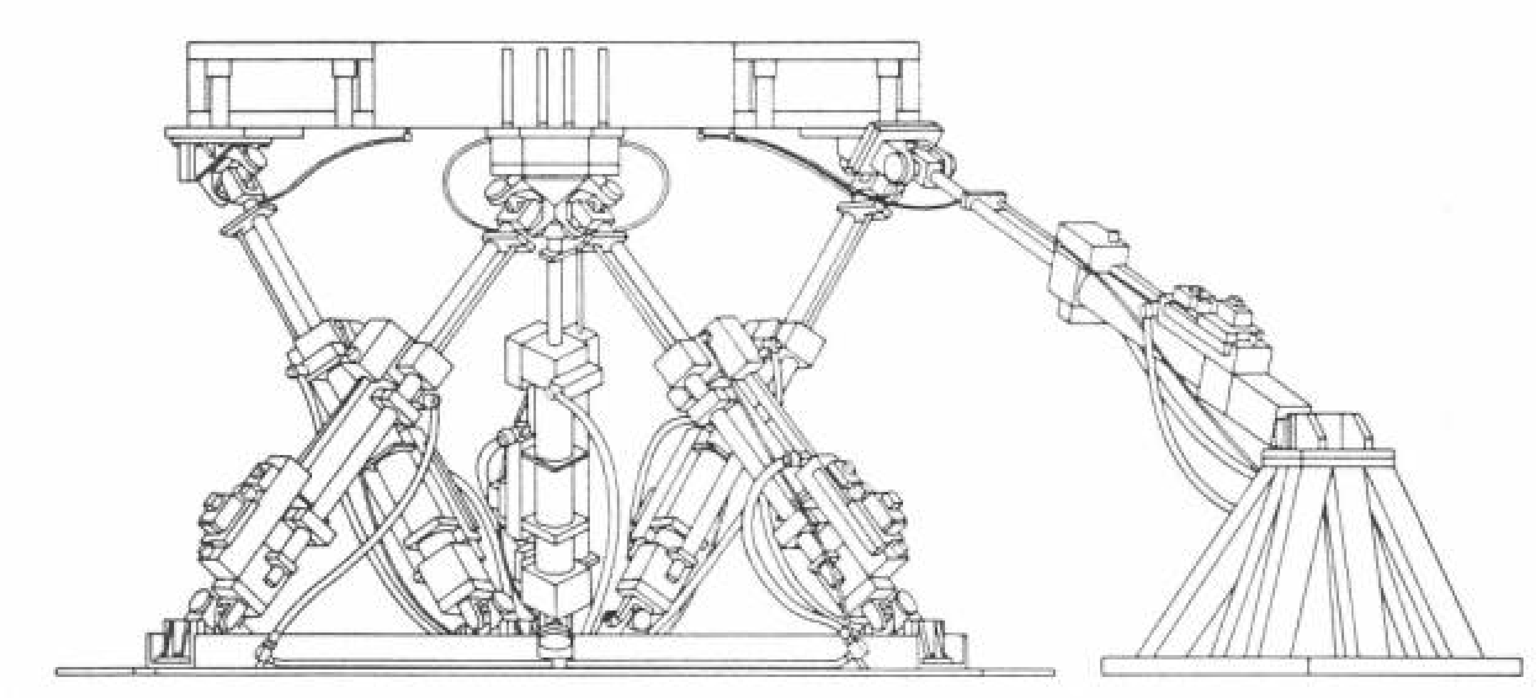
– For example, the spring-loaded relief valve is completely shut until pressure reaches the low end of the regulating range, whereupon it opens sufficiently to bypass any pump flow not required by the servovalve.
– The fluid power of the bypassed flow is completely lost and converted to heat.



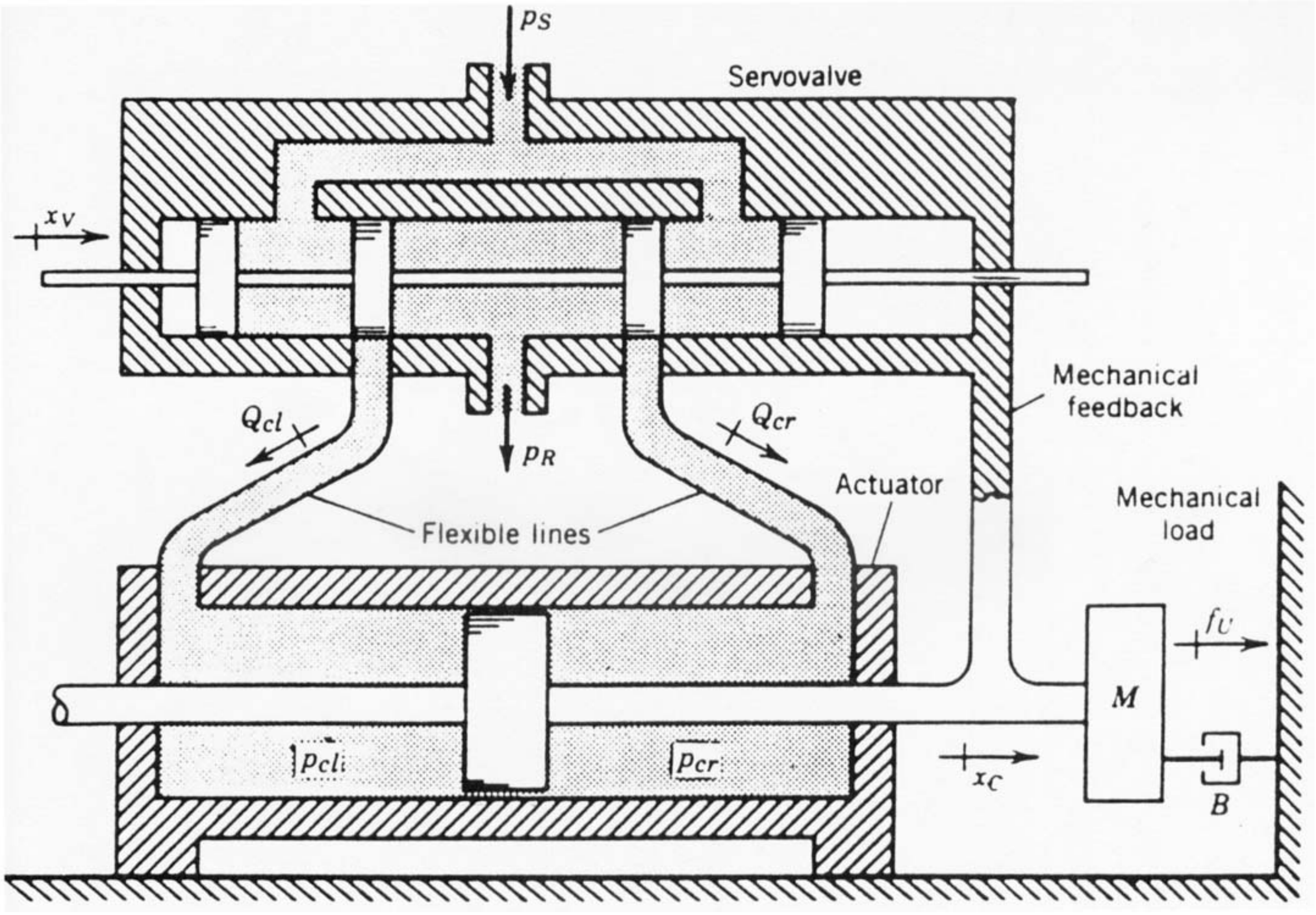
pump flow

- When the servo system requires no flow, all the pump-generated power is converted to heat, giving zero efficiency.
- Supply pressure p_s varies by about ± 3% over the regulating range for steady conditions and response to transient flow requirements is quite fast.
- Pump size must be chosen to match the largest anticipated servo-system demand.
- Thus the standard assumption of constant p_s used in servo-system analysis is usually reasonable.

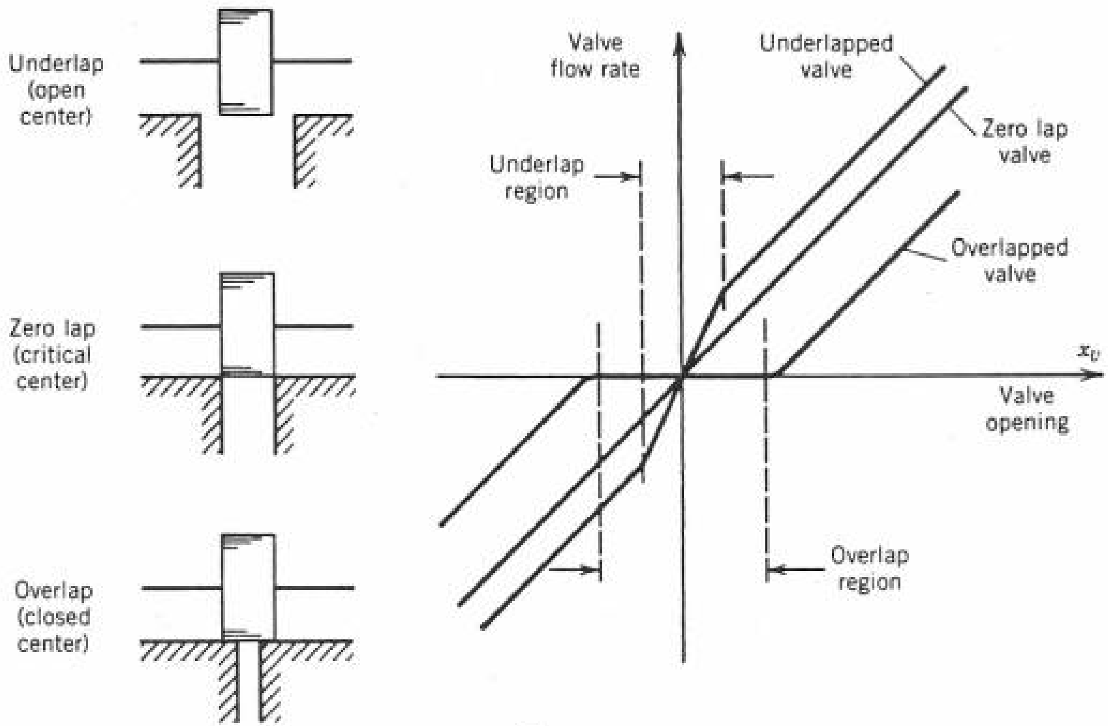
- The figure below shows a flight simulator, a sophisticated application of valve-controlled servos where the motions of six actuators are coordinated to provide roll, pitch, and yaw rotary motions plus x, y, z translation.

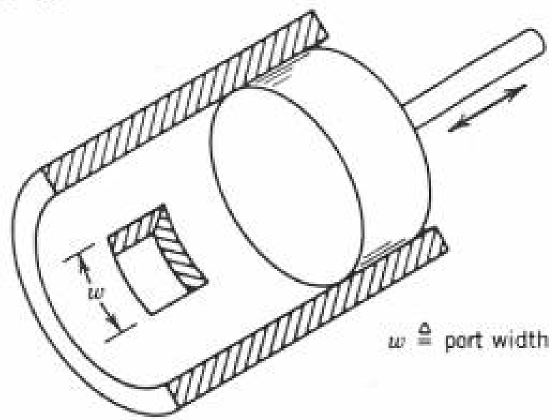


Physical System: Valve-Controlled Servo



Real valves always exhibit either underlap ($x_u > 0$) or overlap ($x_u < 0$) behavior.
Underlap / Overlap effects are usually very small.





spool valve

Physical Modeling Assumptions

- supply pressure is constant at 1000 psig
- reservoir pressure is constant at 0 psig
- valve is zero lap
- actuator pressures p_{cl} and p_{cr} each come to $p_s/2$ at the servo rest condition
- neglect inertia of the fluid
- cylinder and piston are rigid
- sensor dynamics are negligible
- parameters are constant

- compressibility effects are neglected in the orifice flow equations, but not in the cylinder equations as pressures can be high during acceleration and deceleration periods and oil compressibility can have a destabilizing effect
- both flow orifices are identical, i.e., the flow and pressure coefficients are identical for both
- disturbance to the mass is zero

Physical Model Parameters

x_u	0 inches, zero lap condition
p_s	1000 psig (constant), supply pressure
C_d	0.6, orifice discharge coefficient
w	0.5 in, valve port width
ρ	7.8E-5 lbf-s^2/in^4, fluid density
A_p	2.0 in^2, piston area
β	100,000 psi, bulk modulus of fluid
M	0.03 lbf-s^2/in, mass
K_{pl}	0.001 in^3/s-psi, piston leakage coefficient
B	100 lbf-s/in, viscous damping coefficient
V_{l0}	4.0 in^3, volume at operating point of left cylinder
V_{r0}	4.0 in^3, volume at operating point of right cylinder
p_{cl0}	500 psi, initial pressure of left cylinder
p_{cr0}	500 psi, initial pressure of right cylinder
x_{C0}	0 in, initial displacement of mass
$\dot{x}_{C0}$	0 in/sec, initial velocity of mass
f_U	0 lbf, disturbance
p_r	0 psig, return pressure

Nonlinear Mathematical Model

- Equations for the orifice volume flow rates Q_{cl} and Q_{cr} for the left and right ends of the cylinder
- Equations for conservation of mass (continuity equation) for the left and right ends of the cylinder
- Newton's 2^{nd} Law applied to the moving mass
- Kinematic relation representing the mechanical feedback

The Variable Orifice

- The variable orifice is at the heart of most fluid power systems and is the most popular device for controlling fluid flow. It is the fluid equivalent of the electrical resistor and like the resistor its use leads to energy dissipation.
- Overriding advantages include simplicity, reliability, and ease of manufacture.
- The orifice can be used in analog (infinite number of positions) or discontinuous (fully open or fully closed) modes, depending on the application.

- Knowledge of the orifice equations for incompressible and compressible flow is essential. Here we consider incompressible flow.
- The orifice equation for the volume flow rate Q of an incompressible fluid, assuming that the upstream area is much greater than the orifice area A, is:

$$Q = C_d A \sqrt{\frac{2\left(P_u - P_d\right)}{\rho}} \qquad C_d = \text{orifice discharge coefficient}$$

- In a given system the dominating variables are usually the pressure drop and the orifice area.

- The predominant nonlinearity is the square root term, but C_d depends on the Reynolds number and cavitation.
- Cavitation refers to the formation and collapse of cavities, containing air or gas, in the liquid. If the pressure is reduced far enough hydraulic oil will vaporize and vapor cavities will form. The pressure at which vaporization commences is called the vapor pressure of the liquid and is very dependent on the temperature of the liquid. As the temperature increases, the vapor pressure increases.

- The phenomenon of cavitation damage in hydraulic machinery, turbines, pumps, and propellers is well known. It has been shown both analytically and experimentally that when cavities collapse as a result of increased hydraulic pressure, very large pressures can be developed. However, there is still controversy about the exact mechanism of the damaging process.

Orifice Flow-Rate Equations

x_u positive: valve underlap
x_u negative: valve overlap
x_u zero: valve zero lap
x_v is displacement of valve spool

$$Q_{cl} = C_d w(x_u + x_v)\sqrt{\frac{2(p_s - p_{cl})}{\rho}}$$ valid when $(x_u + x_v)$ is $>$ 0.

This is flow into the left cylinder.

$$Q_{cl} = -C_d w(x_u - x_v)\sqrt{\frac{2(p_{cl} - p_r)}{\rho}}$$ valid when $(x_u - x_v)$ is $>$ 0.

This is flow out of the left cylinder.

$$Q_{cr} = -C_d w(x_u + x_v)\sqrt{\frac{2(p_{cr} - p_r)}{\rho}}$$ valid when $(x_u + x_v)$ is $>$ 0.

This is flow out of the right cylinder.

$$Q_{cr} = C_d w(x_u - x_v)\sqrt{\frac{2(p_s - p_{cr})}{\rho}}$$ valid when $(x_u - x_v)$ is $>$ 0.

This is flow into the right cylinder.

Conservation of Mass Equations

- Conservation of Mass

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho \vec{v} \bullet \overrightarrow{dA}$$

$$0 = \rho_{CV}\dot{V}_{CV} + V_{CV}\dot{\rho}_{CV} + \rho Q_{net}$$

$$0 = \dot{V}_{CV} + \frac{V_{CV}}{\rho}\dot{\rho} + Q_{net}$$

- Here we assume that all of the densities of the system (inlet flow, outlet flow, and control volume) are the same and equal to ρ.

– This assumption is justified for incompressible fluids and is quite accurate for compressible fluids if pressure variations are not too large and the temperature of flow into the control volume is almost equal to the temperature of the flow out of the control volume.

- The isothermal bulk modulus is given by:

$$\beta = \rho_0 \left.\frac{\partial P}{\partial \rho}\right|_{P_0, T_0} = \left.\frac{\partial P}{\partial \rho / \rho_0}\right|_{P_0, T_0}$$

- Therefore:

$$\dot{\rho} = \frac{\partial \rho}{\partial P}\frac{dP}{dt} = \frac{\rho_0}{\beta}\dot{P}$$

- Conservation of Mass can be written as:

$$0 = \dot{V} + \frac{V}{\beta}\dot{P}_{CV} + Q_{net}$$

- Evaluating terms:

$$\text{Left cylinder} \quad \begin{cases} Q_{net} = -Q_{cl} + K_{pl}(p_{cl} - p_{cr}) \\ \dot{V} = A_p \dfrac{dx_C}{dt} \\ \dfrac{V}{\beta}\dot{P}_{CV} = \dfrac{(V_{l0} + A_p x_C)}{\beta}\dfrac{dp_{cl}}{dt} \end{cases}$$

- The resulting equations for the left and right cylinders are:

$$Q_{cr} - \frac{(V_{r0} - A_p x_C)}{\beta}\frac{dp_{cr}}{dt} + K_{pl}(p_{cl} - p_{cr}) = -A_p \frac{dx_C}{dt}$$

$$Q_{cl} - \frac{(V_{l0} + A_p x_C)}{\beta}\frac{dp_{cl}}{dt} - K_{pl}(p_{cl} - p_{cr}) = A_p \frac{dx_C}{dt}$$

- V_{l0} and V_{r0} are the volumes at the operating point of the left and right cylinders, respectively.
- β is the bulk modulus (isothermal) of the fluid defined by the expression:

$$\beta = -V\frac{dp}{dV}$$

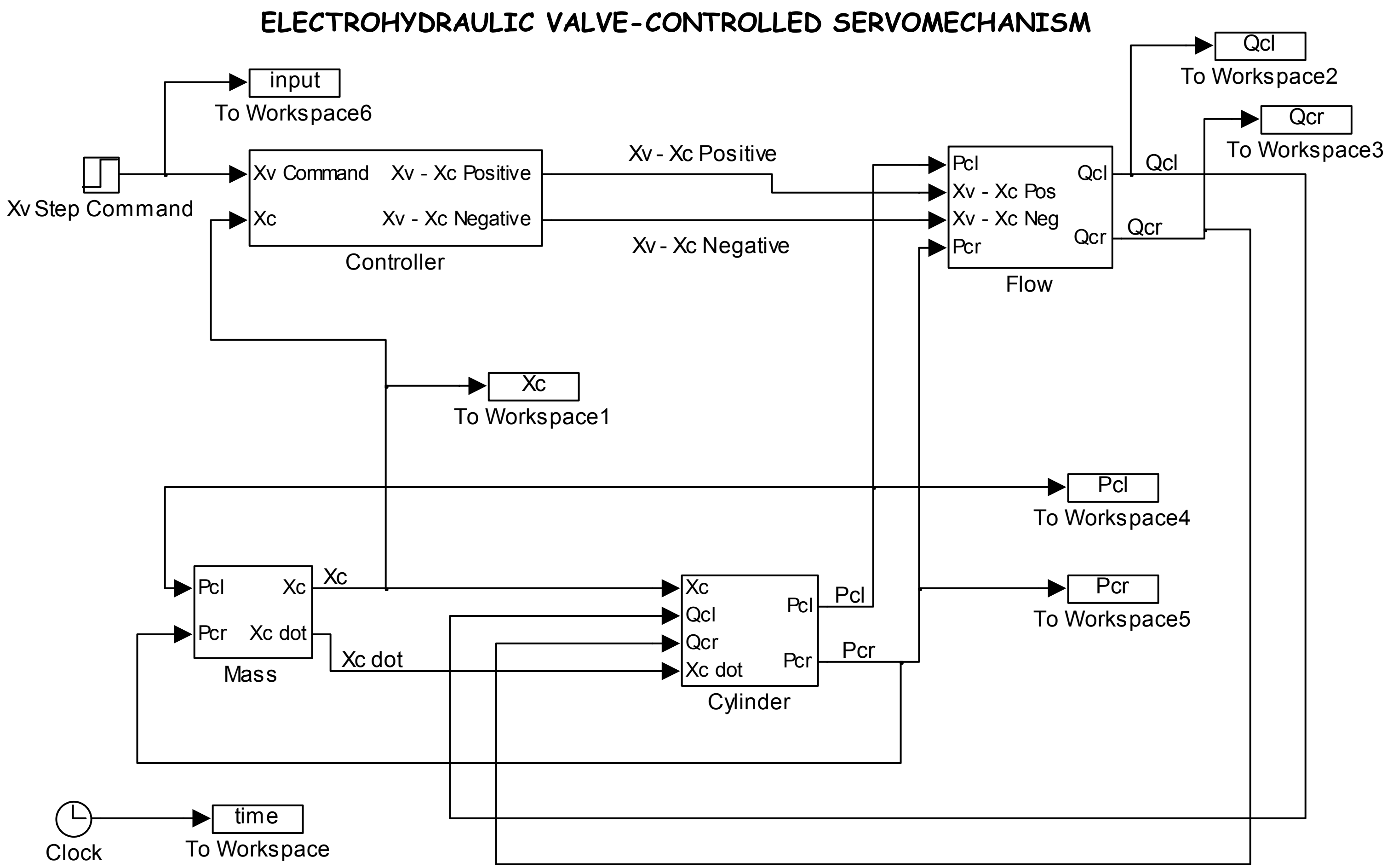
- $(V_{l0} + A_p x_C)$ and $(V_{r0} - A_p x_C)$ represent the compressed volumes of the left and right sides of the cylinder, respectively, which include the lines from the valve to the actuator plus the ends of the cylinder.

Newton's Second Law & Feedback Equations

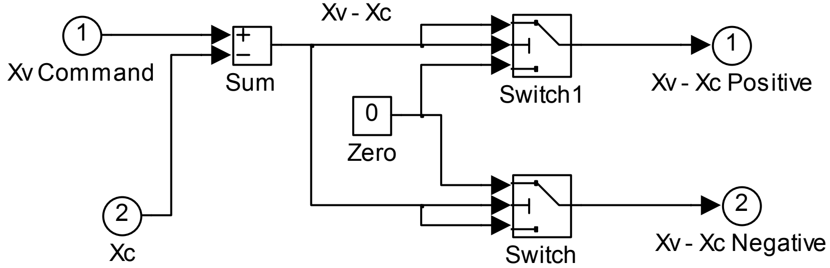
$$(p_{cl} - p_{cr})A_p - B\frac{dx_C}{dt} + f_U = M\frac{d^2x_C}{dt^2}$$

$$x_v = x_V - x_C$$

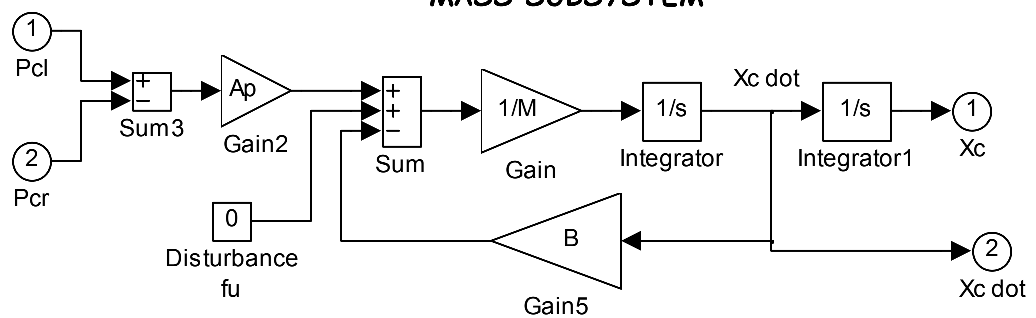
Simulink Block Diagrams

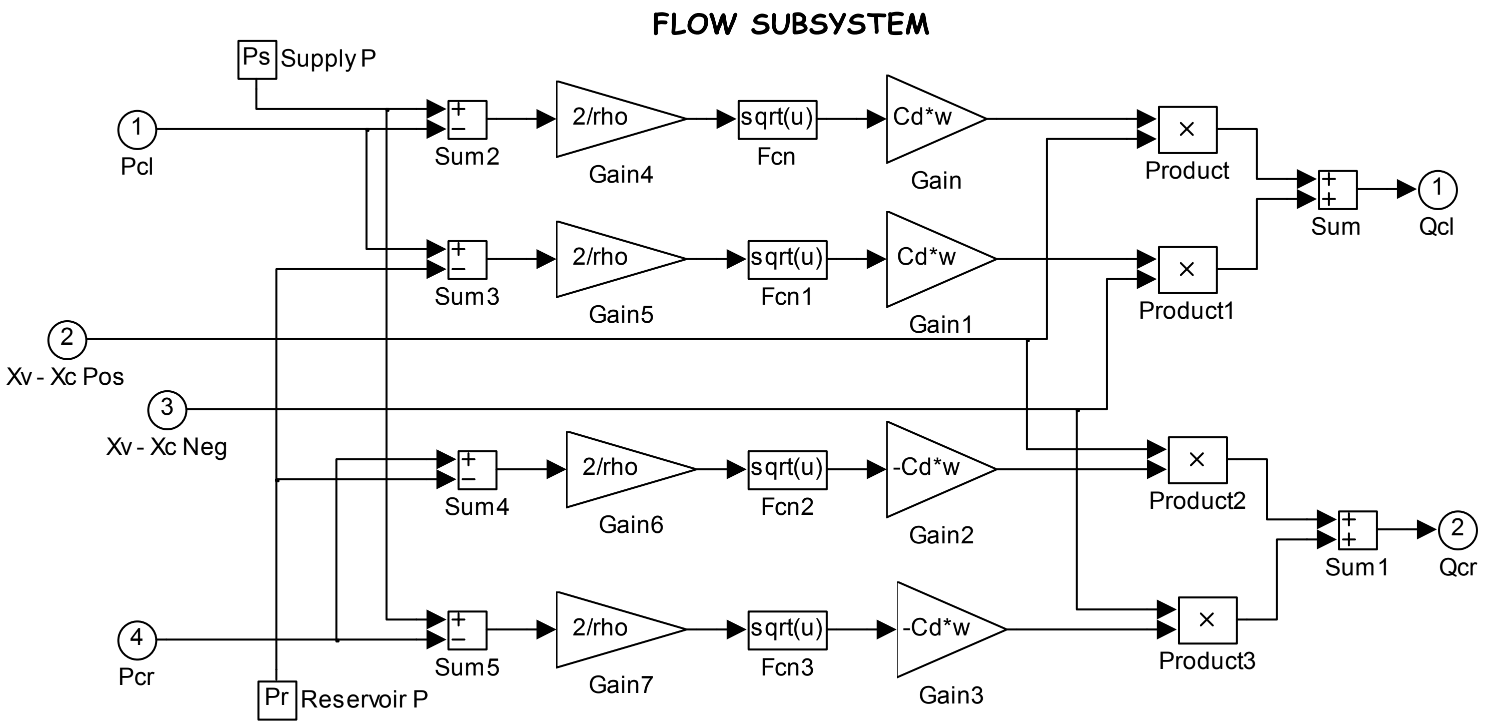


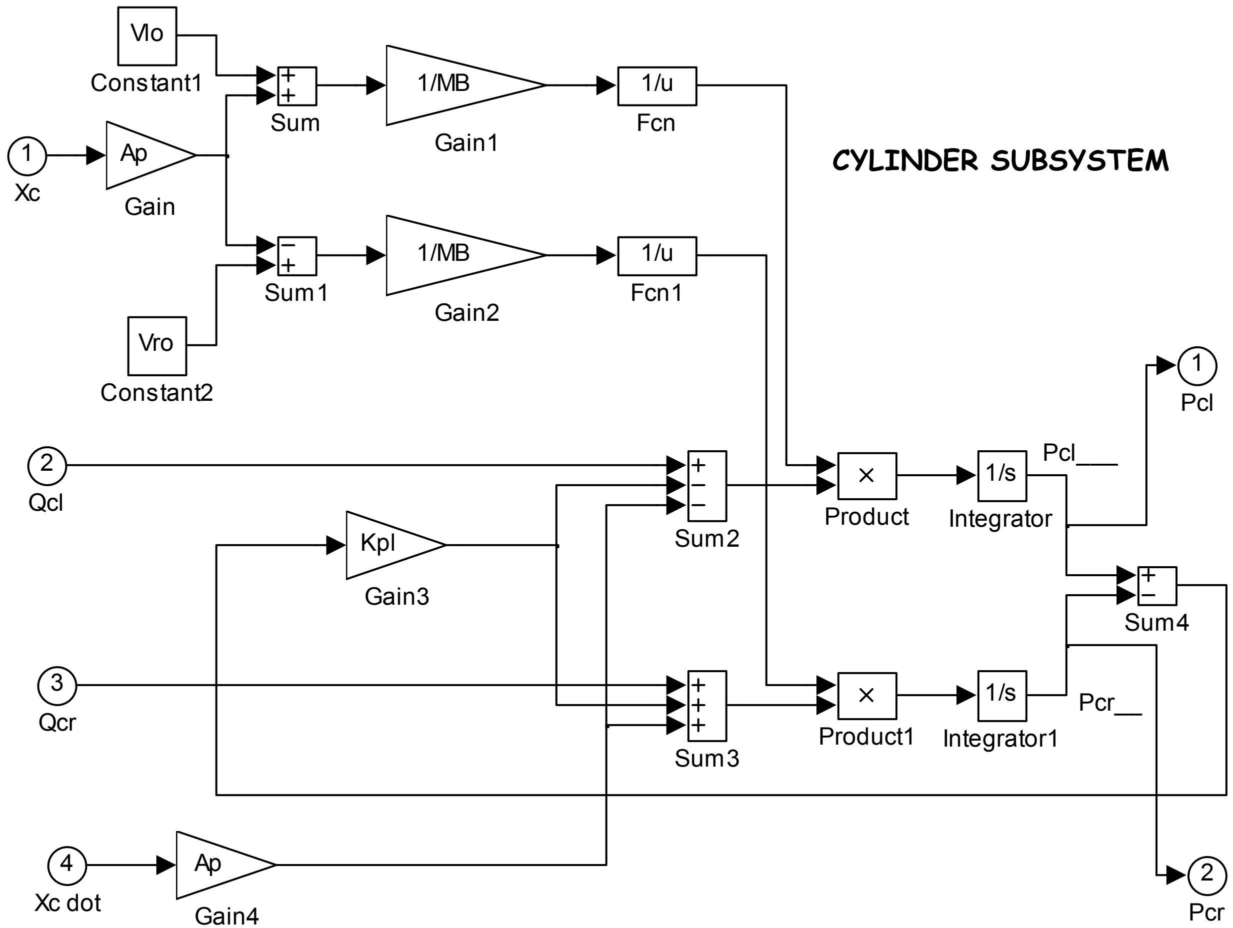
CONTROLLER SUBSYSTEM



MASS SUBSYSTEM




FLOW SUBSYSTEM
Ps
Supply P
1
Pcl
+
–
Sum2
2/rho
Gain4
sqrt(u)
Fcn
Cd*w
Gain
×
Product
+
+
Sum
1
Qcl
Sum3
Gain5
Fcn1
Gain1
Product1
2
Xv - Xc Pos
3
Xv - Xc Neg
Sum4
Gain6
Fcn2
-Cd*w
Gain2
Product2
Sum1
2
Qcr
4
Pcr
Sum5
Gain7
Fcn3
Gain3
Product3
Pr
Reservoir P



CYLINDER SUBSYSTEM
Vlo
Constant1
Sum
1/MB
Gain1
1/u
Fcn
1
Xc
Ap
Gain
Sum1
1/MB
Gain2
1/u
Fcn1
Vro
Constant2
1
Pcl
2
Qcl
Sum2
Product
1/s
Integrator
Pcl___
Kpl
Gain3
Sum4
3
Qcr
Sum3
Product1
1/s
Integrator1
Pcr__
4
Xc dot
Ap
Gain4
2
Pcr

MatLab File of Constants and Expressions

```
M=0.03;
B=100;
Ap=2.0;
Kpl=0.001;
Vlo=4.0;
Vro=4.0;
MB=100000;
Pr=0;
Ps=1000;
rho=7.8e-5;
Cd=0.6;
w=0.5;
Pclo=500;
Pcro=500;
Xcdoto=0;
Xco=0;
Vo=4.0;
Cx=1074.172;
Cp=0;
```

```
A=[-(Cp+Kpl)*MB/Vo Kpl* MB /Vo 0 -Ap* MB/Vo;
Kpl*MB/Vo -(Cp+Kpl)* MB/Vo 0 Ap* MB/Vo;
0 0 0 1;Ap/M -Ap/M 0 -B/M];
B1=[MB*Cx/Vo 0;- MB*Cx/Vo 0;0 0;0 1/M];
C=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
D=[0 0;0 0;0 0;0 0];
K=(2*Cx*Ap)/(2*Ap^2+B*(Cp+2*Kpl));
omegan=sqrt((MB*(2*Ap^2+B*(Cp+2*Kpl)))/(M*Vo));
zeta=(B+((2*MB*M)/Vo)*Kpl+((MB*M)/Vo)*Cp)/(2*sqrt(((MB*M)/Vo)
*(2*Ap^2+B*(Cp+2*Kpl))));
XcXvNum=K*omegan^2;
XcXvDen=[1 2*zeta*omegan omegan^2 0];
%[XcXvNumCL,XcXvDenCL]=cloop(XcXvNum,XcXvDen,-1);
Kc=1;
XcXvNumCL=Kc*K*omegan^2;
XcXvDenCL=[1 2*zeta*omegan omegan^2 Kc*K*omegan^2];
```

Linear System Analysis

- Restrict the analysis to small perturbations around a chosen operating point. A linearized approximate model may be obtained that provides many useful results.
- Valve flow equations can be thought of as relations between a dependent variable (flow rate) and two independent variables (spool motion and cylinder pressure) and thus can be linearized about any desired operating point.

$$Q_v \approx Q_{v,0} + \left.\frac{\partial Q_v}{\partial x_v}\right|_{\text{operating point}} x_{v,p} + \left.\frac{\partial Q_v}{\partial p_c}\right|_{\text{operating point}} p_{c,p}$$

$$\text{flow gain} = C_x = \left.\frac{\partial Q_v}{\partial x_v}\right|_{\text{operating point}}$$

$$\text{pressure coefficient} = C_p = -\left.\frac{\partial Q_v}{\partial p_c}\right|_{\text{operating point}}$$

$$Q_v \approx Q_{v,0} + C_x x_{v,p} - C_p p_{c,p}$$

- Assume that $Q_{v,0} = 0$ and that the numerical values of C_x and C_p are equal for the Q_{cl} and Q_{cr} equations (correct assumptions for commonly used operating points).

$$Q_{cl} \approx C_x x_{v,p} - C_p p_{cl,p} \qquad Q_{cr} \approx -C_x x_{v,p} - C_p p_{cr,p}$$

- Take the volumes $(V_{l0} + A_p x_C)$ and $(V_{r0} - A_p x_C)$ to be constant at $V_{l0} = V_{r0} = V_0$, a good approximation for small changes in x_C.

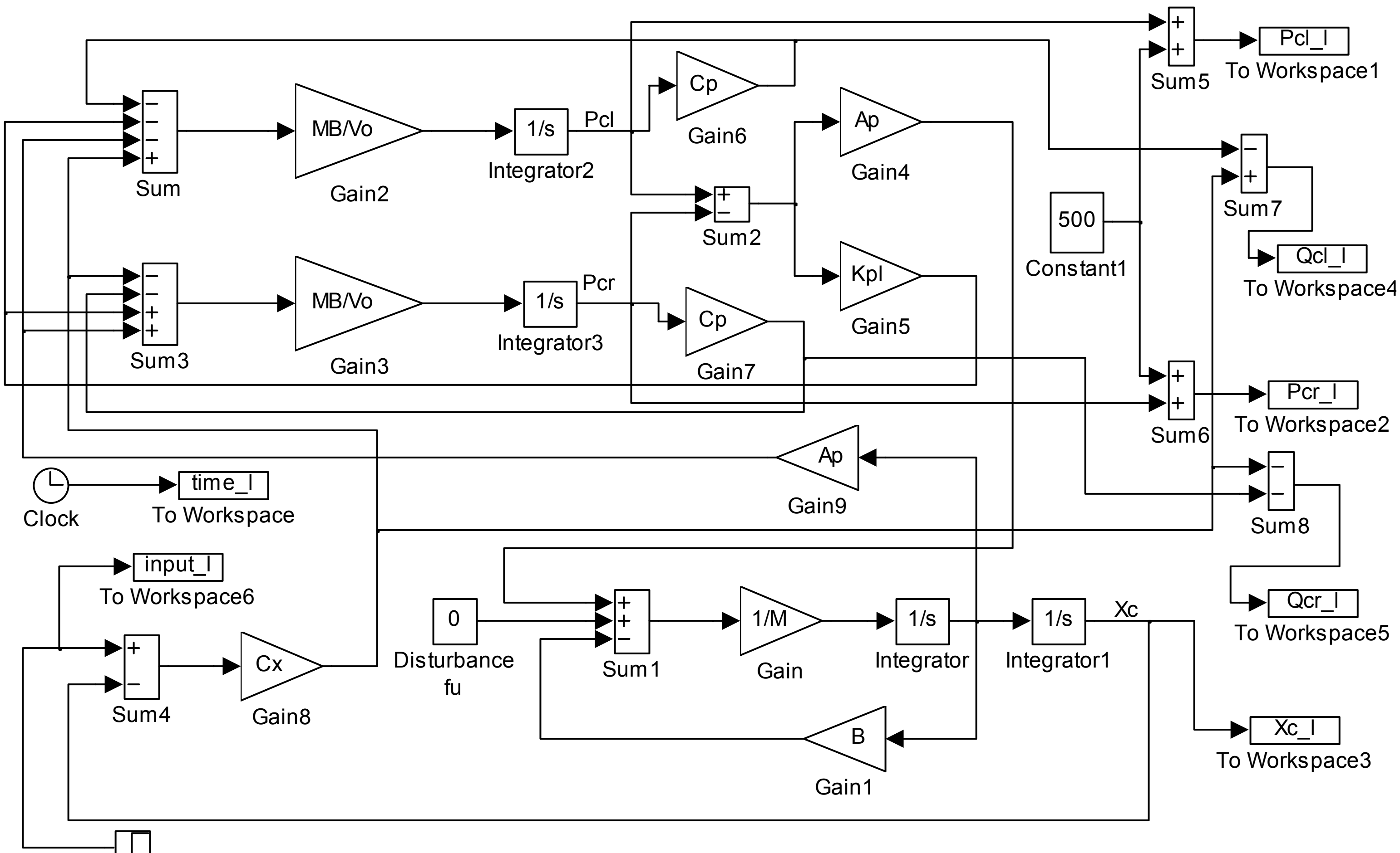
Linearized Set of Equations:

$$\left(C_x x_{v,p} - C_p p_{cl,p}\right) - \frac{V_0}{\beta}\frac{dp_{cl,p}}{dt} - K_{pl}\left(p_{cl,p} - p_{cr,p}\right) = A_p \frac{dx_{C,p}}{dt}$$

$$\left(-C_x x_{v,p} - C_p p_{cr,p}\right) - \frac{V_0}{\beta}\frac{dp_{cr,p}}{dt} + K_{pl}\left(p_{cl,p} - p_{cr,p}\right) = -A_p \frac{dx_{C,p}}{dt}$$

$$\left(p_{cl,p} - p_{cr,p}\right)A_p - B\frac{dx_{C,p}}{dt} + f_{U,p} = M\frac{d^2 x_{C,p}}{dt^2}$$

Simulink Block Diagram of the Linear System



ELECTROHYDRAULIC VALVE-CONTROLLED SERVOMECHANISM (LINEAR)

- If we take the Laplace Transform of these equations, we can derive six useful transfer functions relating the two inputs, x_v and f_U, to the three outputs, p_{cl}, p_{cr}, and x_C.

$$\begin{bmatrix} \dfrac{V_0 s + \beta\left(K_{pl} + C_p\right)}{C_x \beta} & \dfrac{-K_{pl}}{C_x} & \dfrac{A_p s}{C_x} \\ \dfrac{K_{pl}}{C_x} & \dfrac{-V_0 s - \beta\left(K_{pl} + C_p\right)}{C_x \beta} & \dfrac{A_p s}{C_x} \\ -A_p & A_p & Ms^2 + Bs \end{bmatrix} \begin{bmatrix} p_{cl} \\ p_{cr} \\ x_C \end{bmatrix} = \begin{bmatrix} x_v \\ x_v \\ f_U \end{bmatrix}$$

One of these transfer functions is:

$$\frac{x_C}{x_v}(s) = \frac{K}{s\left(\dfrac{s^2}{\omega_n^2} + \dfrac{2\zeta s}{\omega_n} + 1\right)}$$

where

$$K = \frac{2C_x A_p}{2A_p^2 + B\left(C_p + 2K_{pl}\right)}$$

$$\omega_n = \sqrt{\frac{\beta\left[2A_p^2 + B\left(C_p + 2K_{pl}\right)\right]}{MV_0}}$$

$$\zeta = \frac{B + \left(\dfrac{2\beta M}{V_0}\right)K_{pl} + \left(\dfrac{\beta M}{V_0}\right)C_p}{2\sqrt{\dfrac{\beta M}{V_0}\left[2A_p^2 + B\left(C_p + 2K_{pl}\right)\right]}}$$

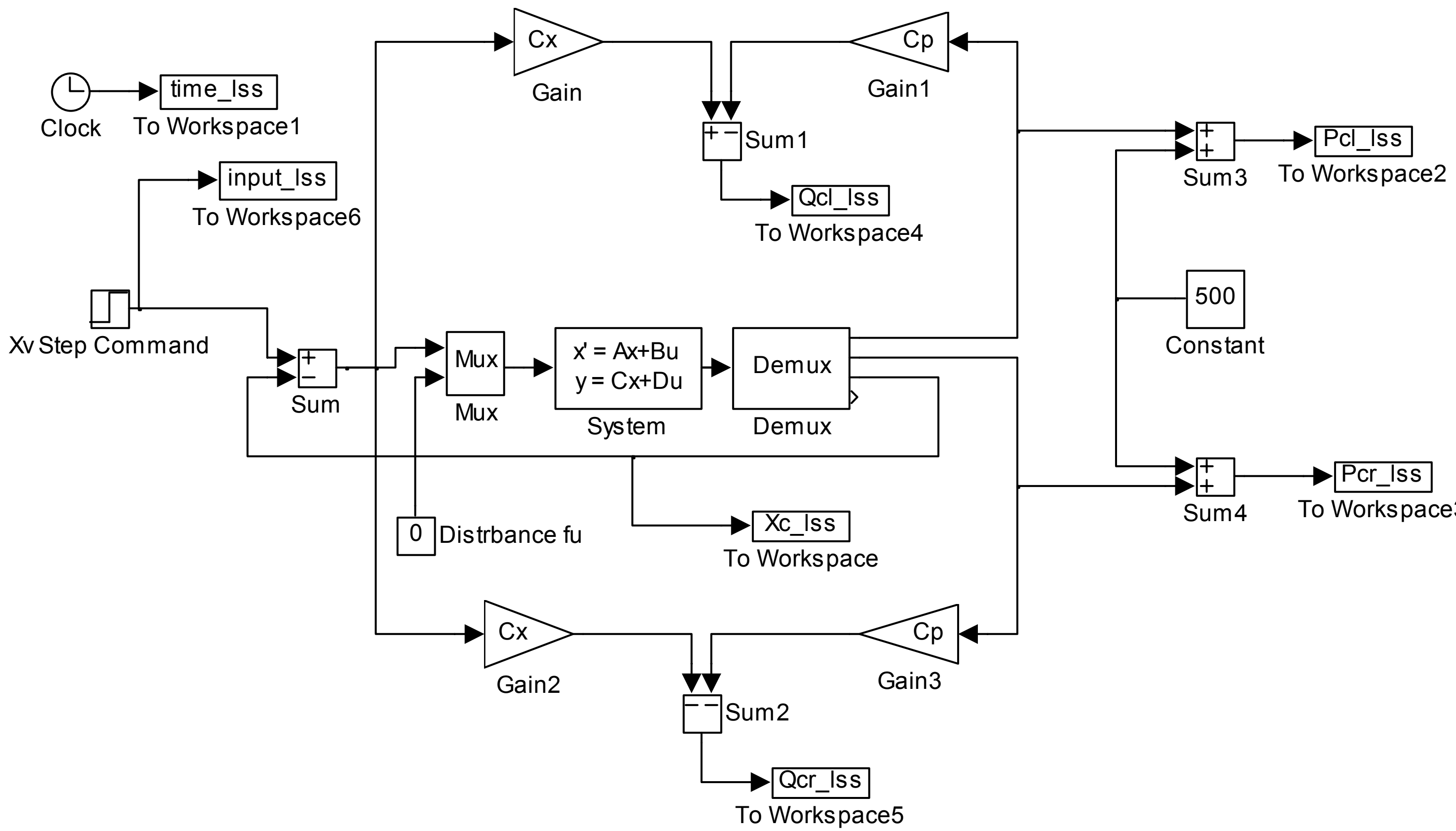
The state variables for the linearized equations are:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} p_{cl} \\ p_{cr} \\ x_C \\ \dot{x}_C \end{bmatrix}$$

The state-variable equations are:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} \dfrac{-\beta\left(C_p + K_{pl}\right)}{V_0} & \dfrac{K_{pl}\beta}{V_0} & 0 & \dfrac{-A_p\beta}{V_0} \\ \dfrac{K_{pl}\beta}{V_0} & \dfrac{-\beta\left(C_p + K_{pl}\right)}{V_0} & 0 & \dfrac{A_p\beta}{V_0} \\ 0 & 0 & 0 & 1 \\ \dfrac{A_p}{M} & \dfrac{-A_p}{M} & 0 & \dfrac{-B}{M} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} \dfrac{\beta C_x}{V_0} \\ \dfrac{-\beta C_x}{V_0} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_v \end{bmatrix}$$

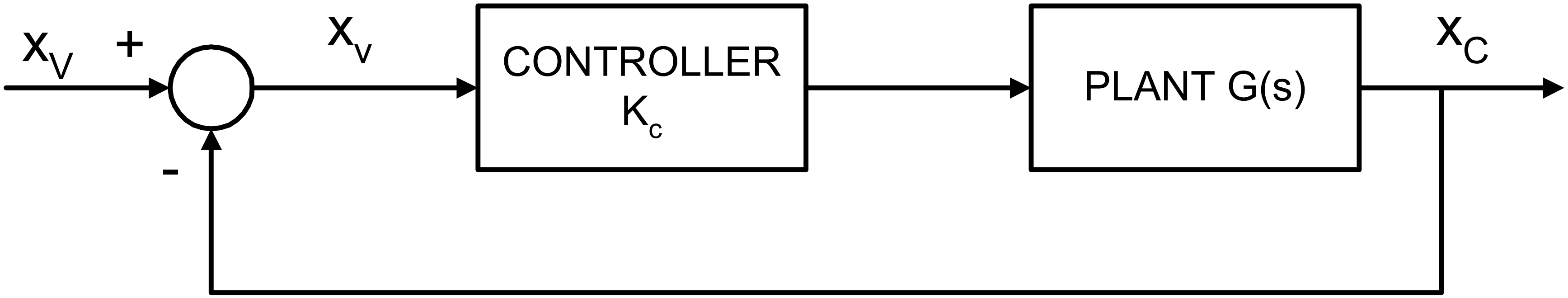
Simulink Block Diagram: Linear System in State-Variable Form



ELECTROHYDRAULIC VALVE-CONTROLLED SERVOMECHANISM (LINEAR - STATE SPACE)

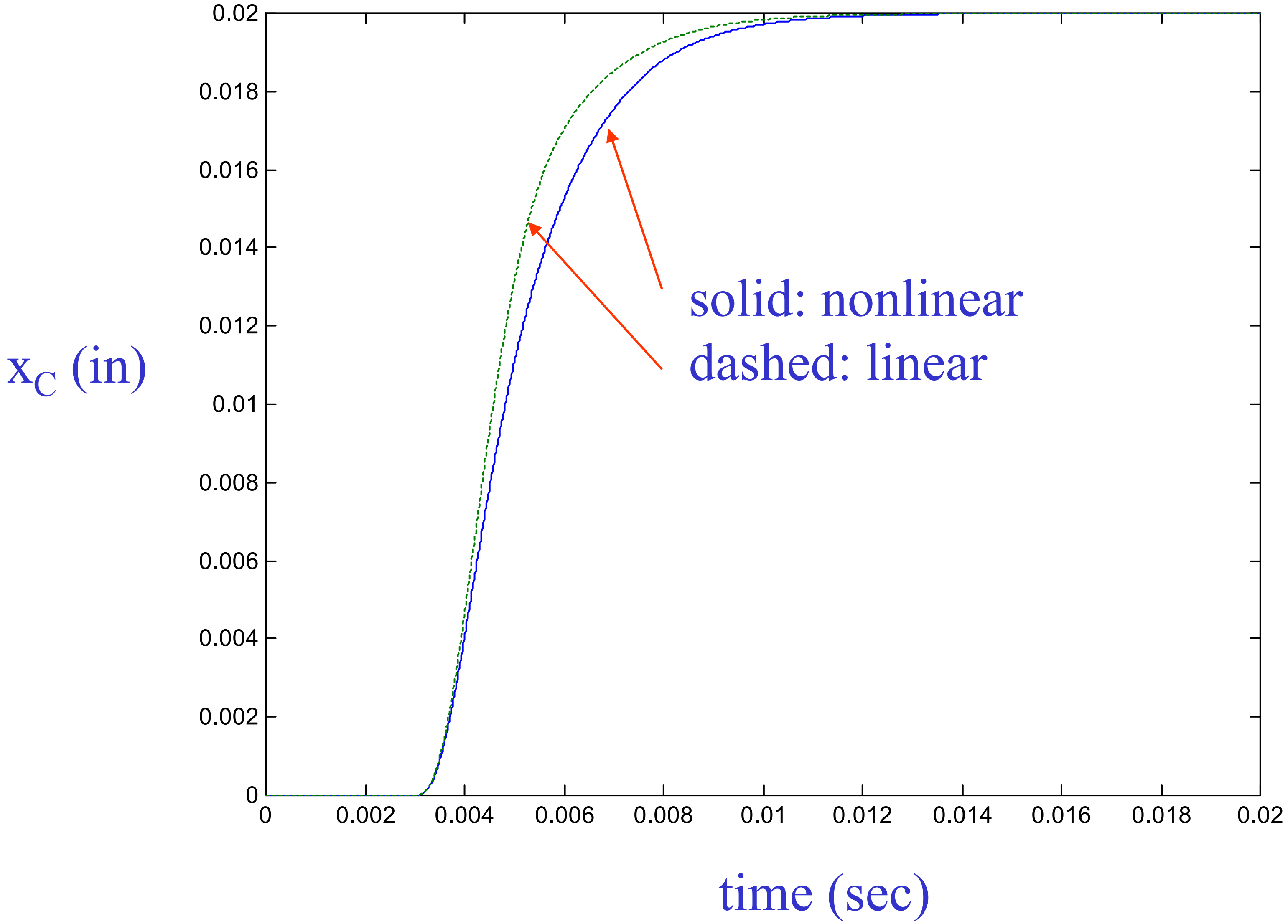
Simulation Results:

Step Command $x_V = 0.02$ in. applied at $t = 0.003$ sec

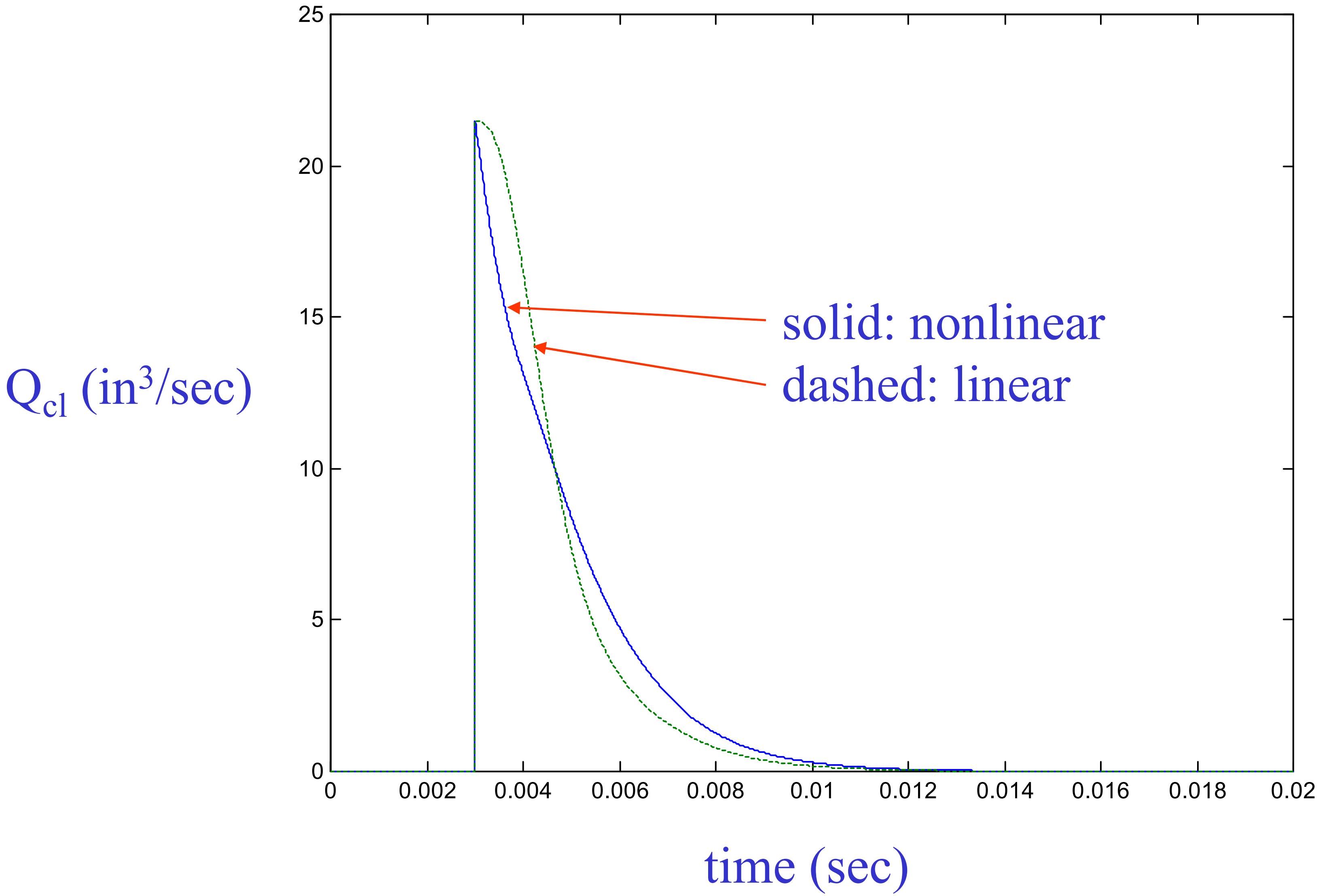


$$\frac{x_C}{x_V} = \frac{K_c G(s)}{1 + K_c G(s)} = \frac{K_c K \omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_c K \omega_n^2}$$

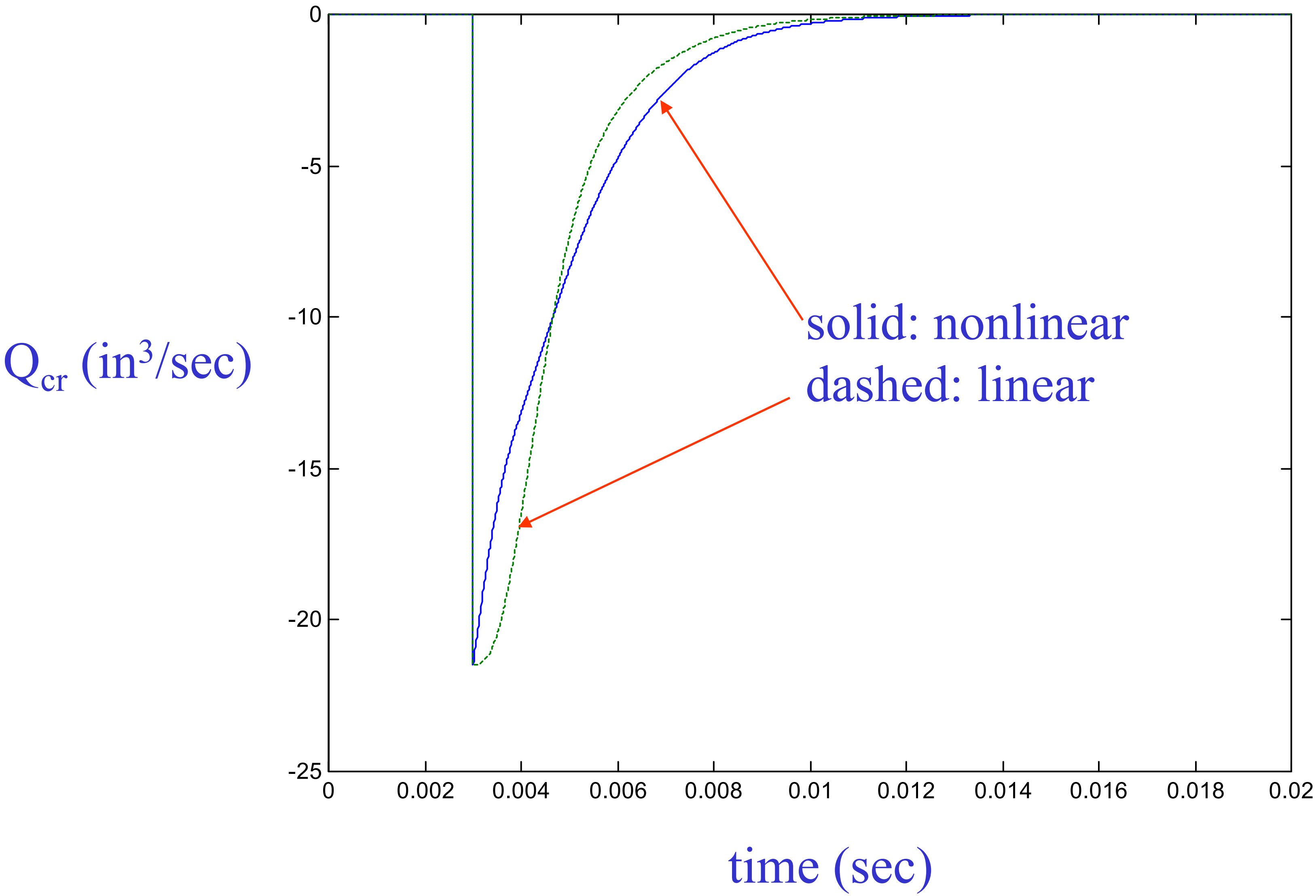
Nonlinear and Linear Simulation Results: x_C vs. time



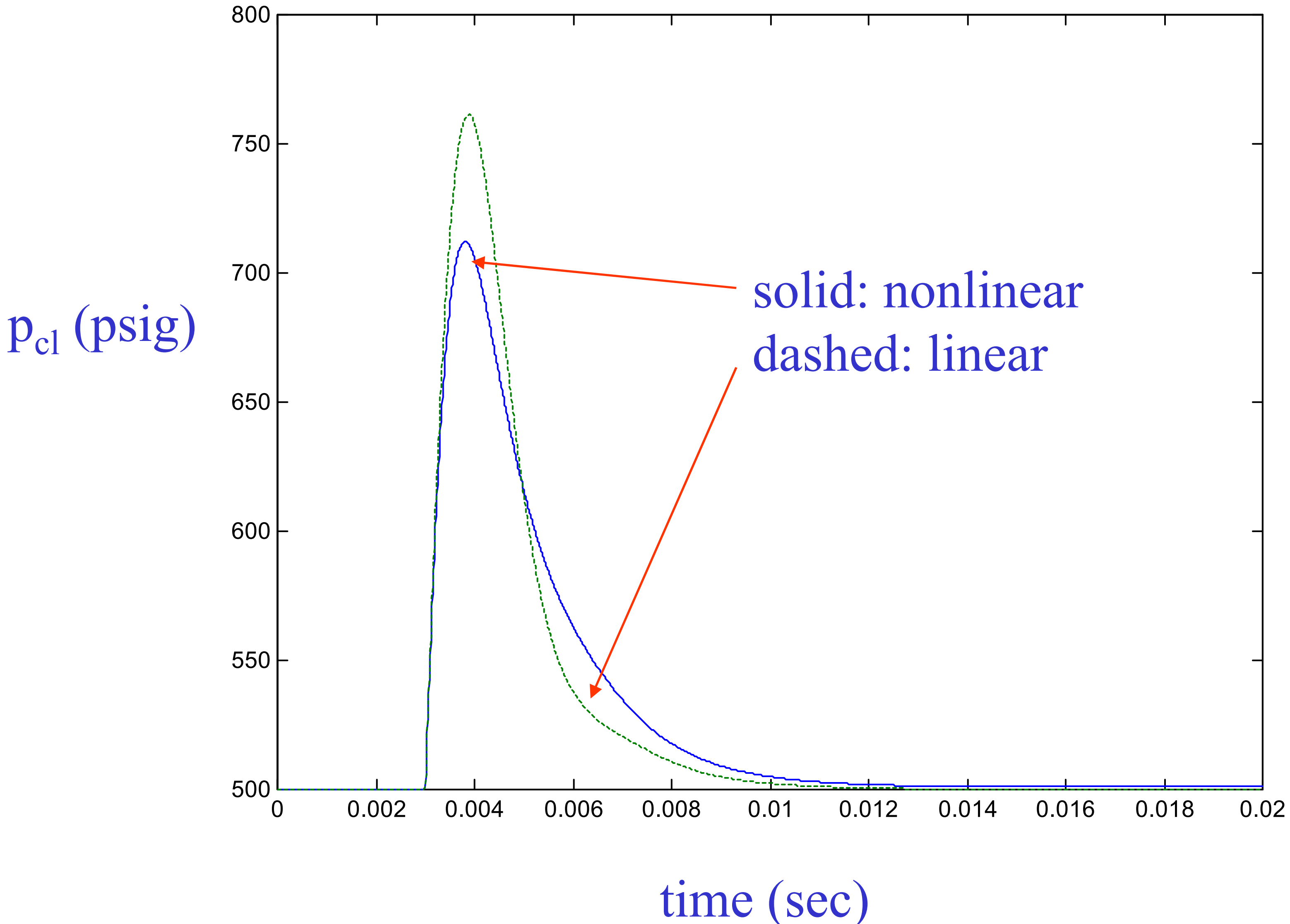
Nonlinear and Linear Simulation Results: Q_{cl} vs. time



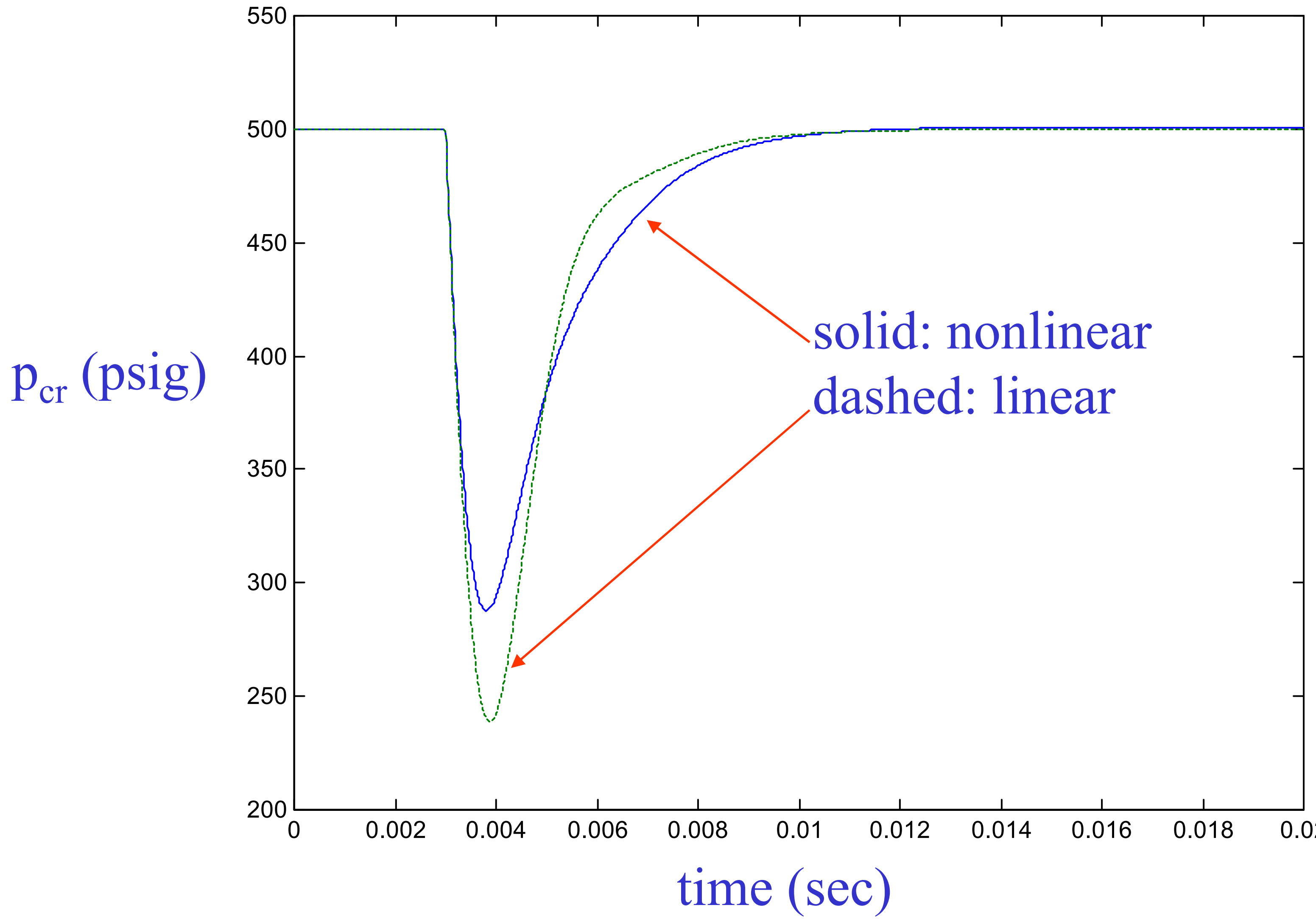
Nonlinear and Linear Simulation Results: Q_{cr} vs. time



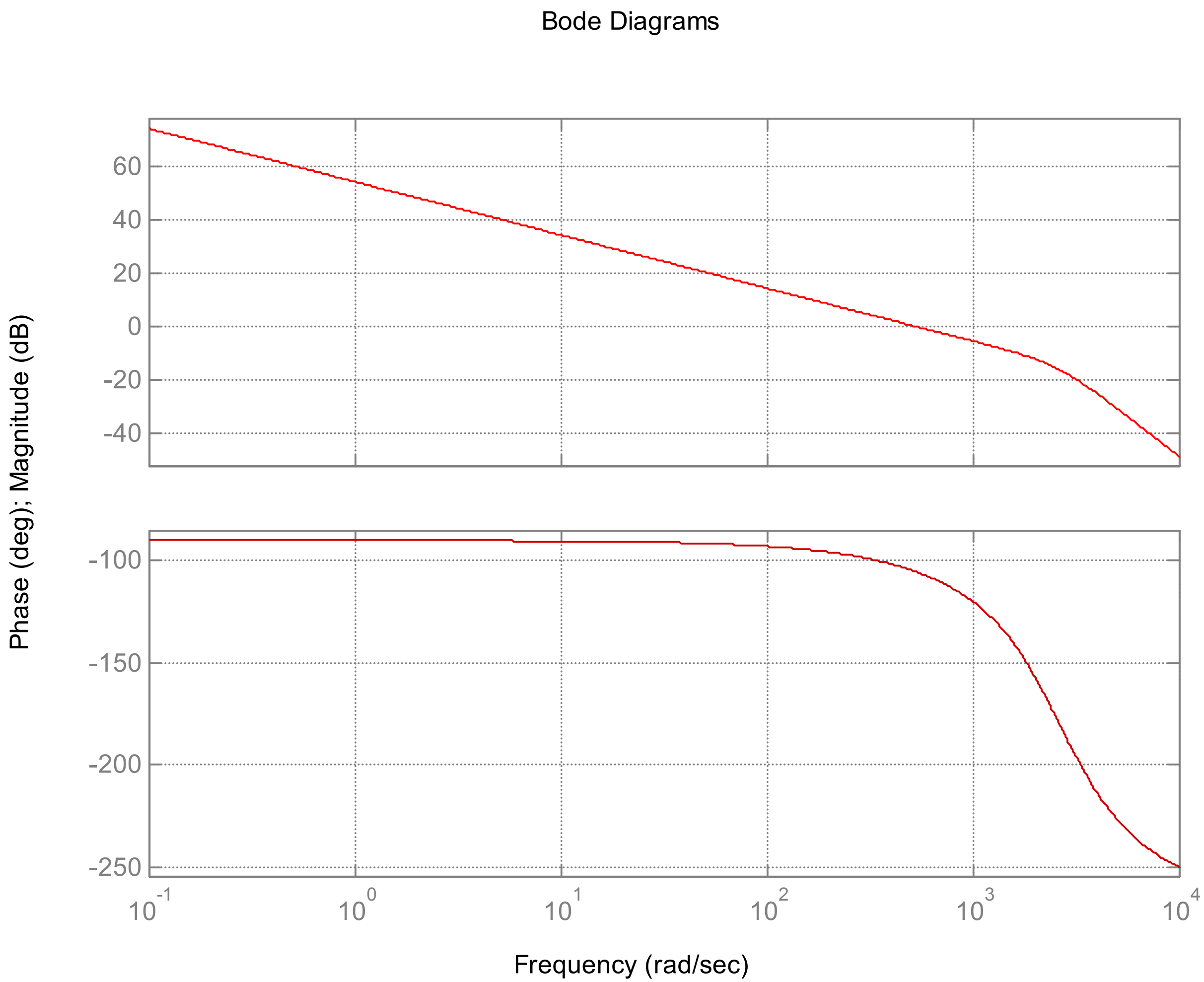
Nonlinear and Linear Simulation Results: p_{cl} time



Nonlinear and Linear Simulation Results: p_{cr} vs. time

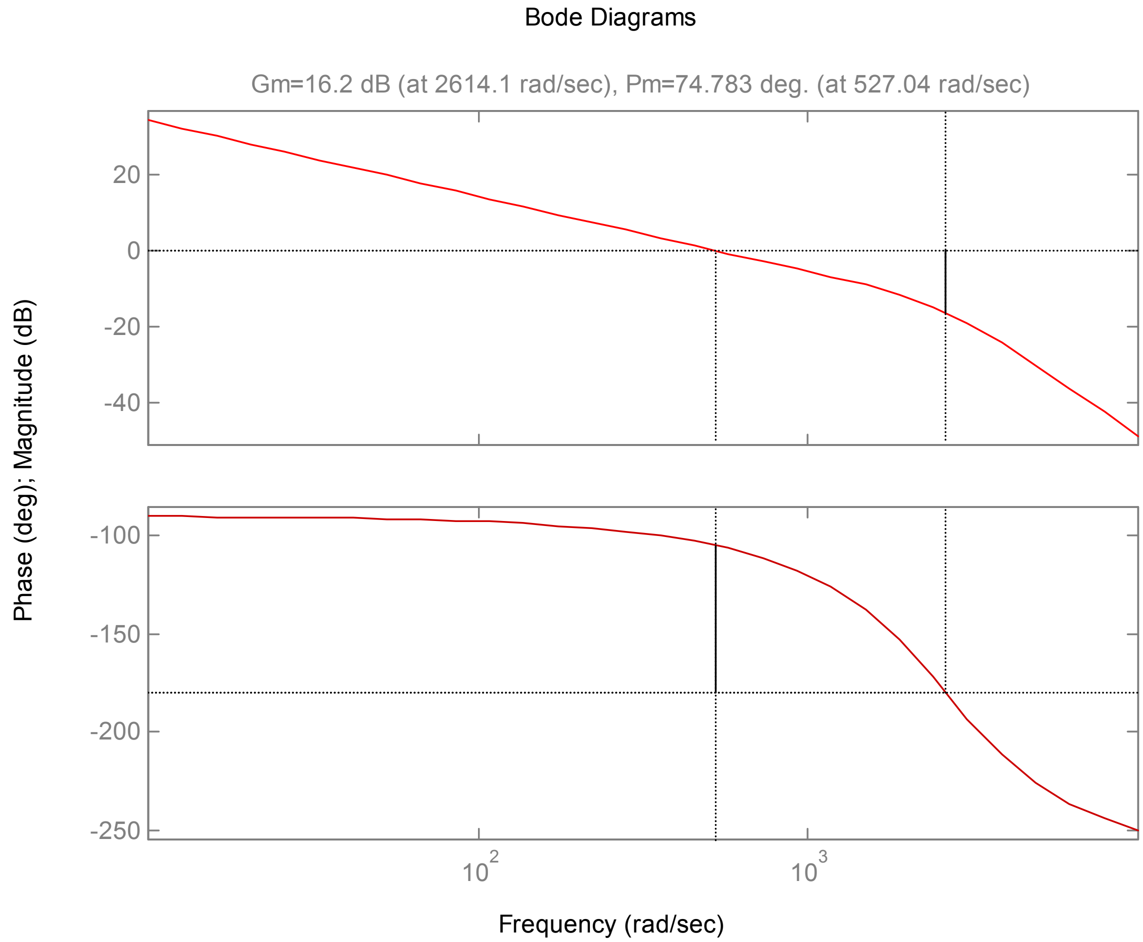


Open-Loop Frequency Response Plots with $K_c = 1$



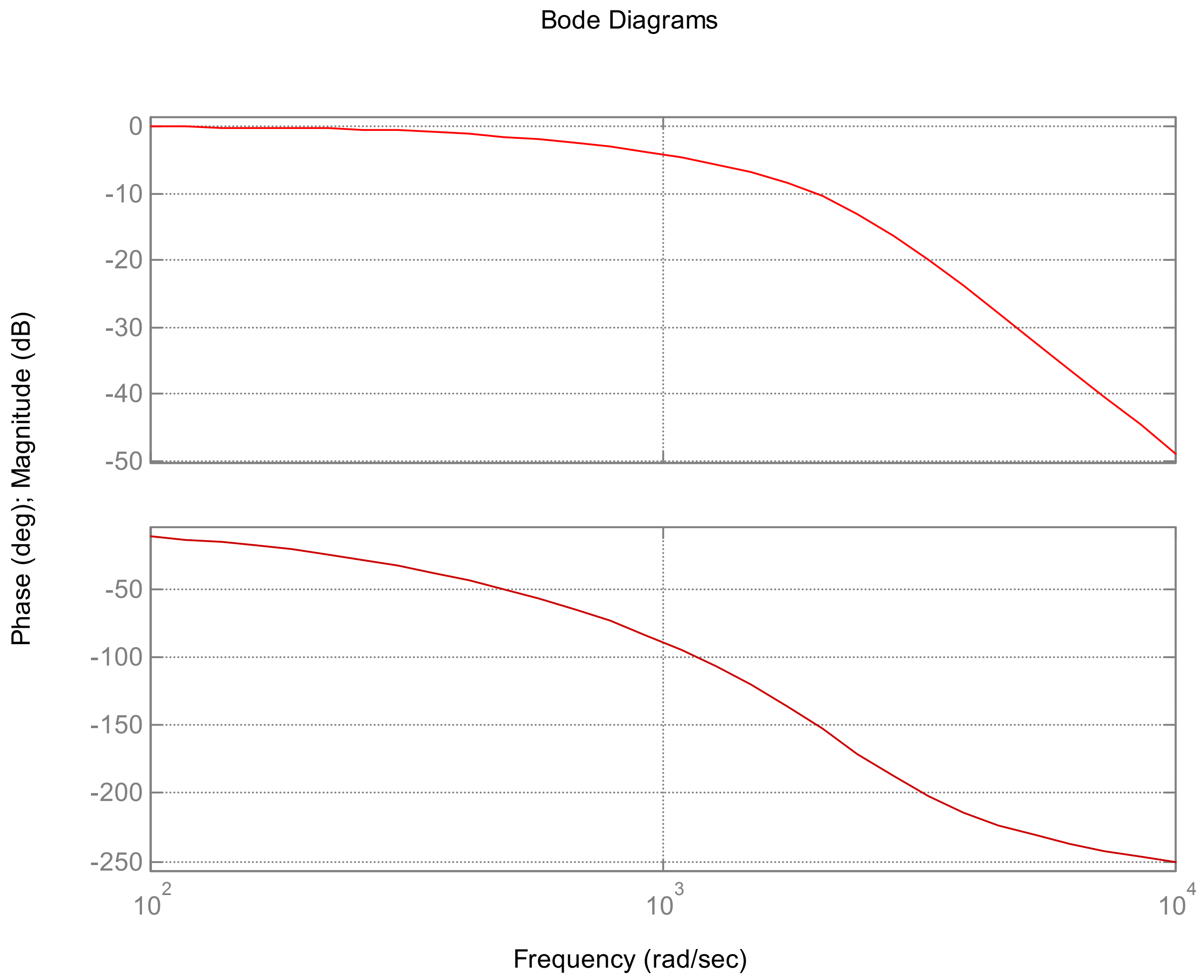
GM = 16.2 dB = 6.46

PM = 74.8°

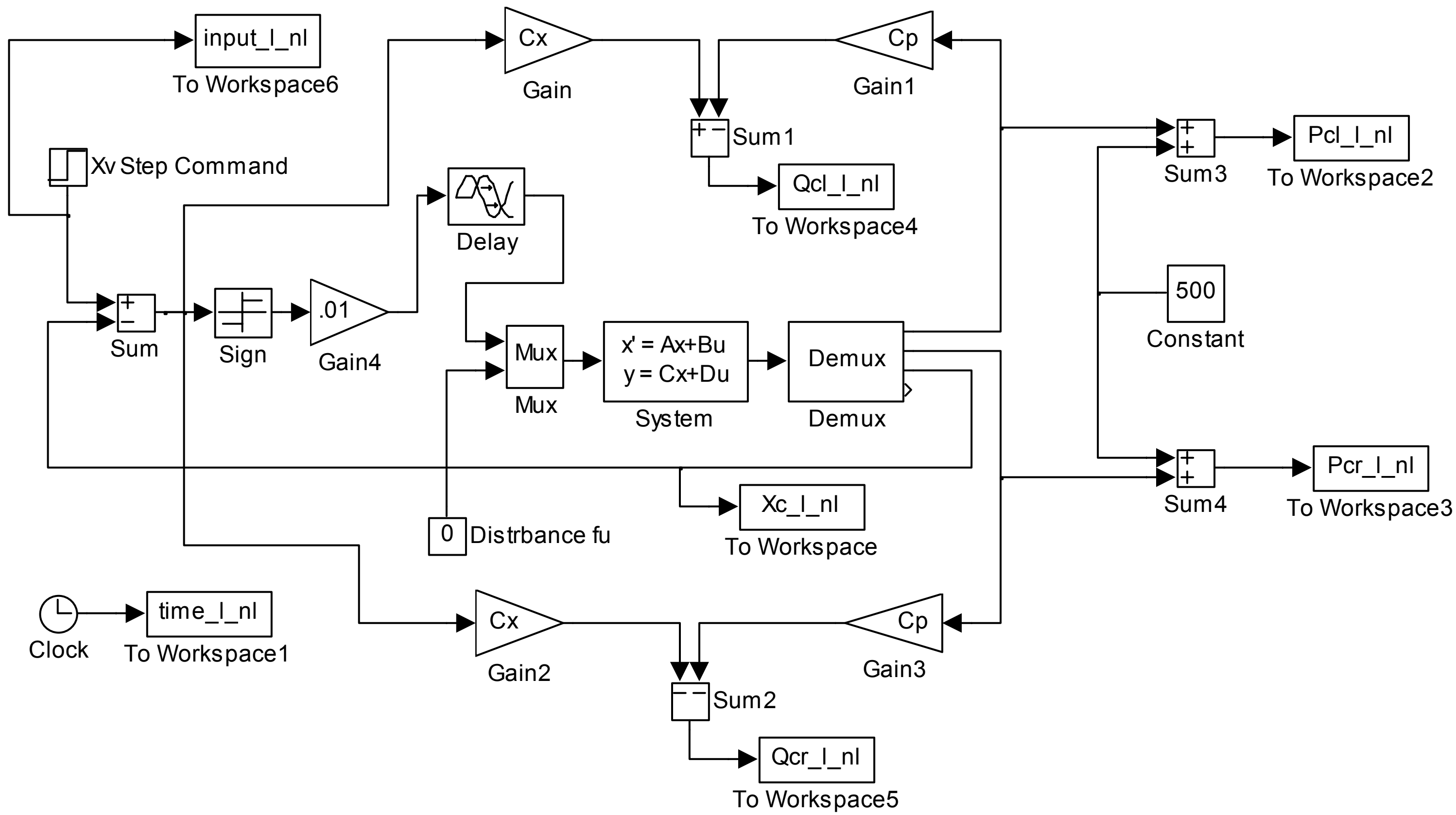


Closed-Loop Frequency Response Plots with $K_c = 1$ Closed-Loop Bandwidth = 123 Hz = 774 rad/sec

At 774 rad/sec:
Mag = 0.707
Phase = -72.2°

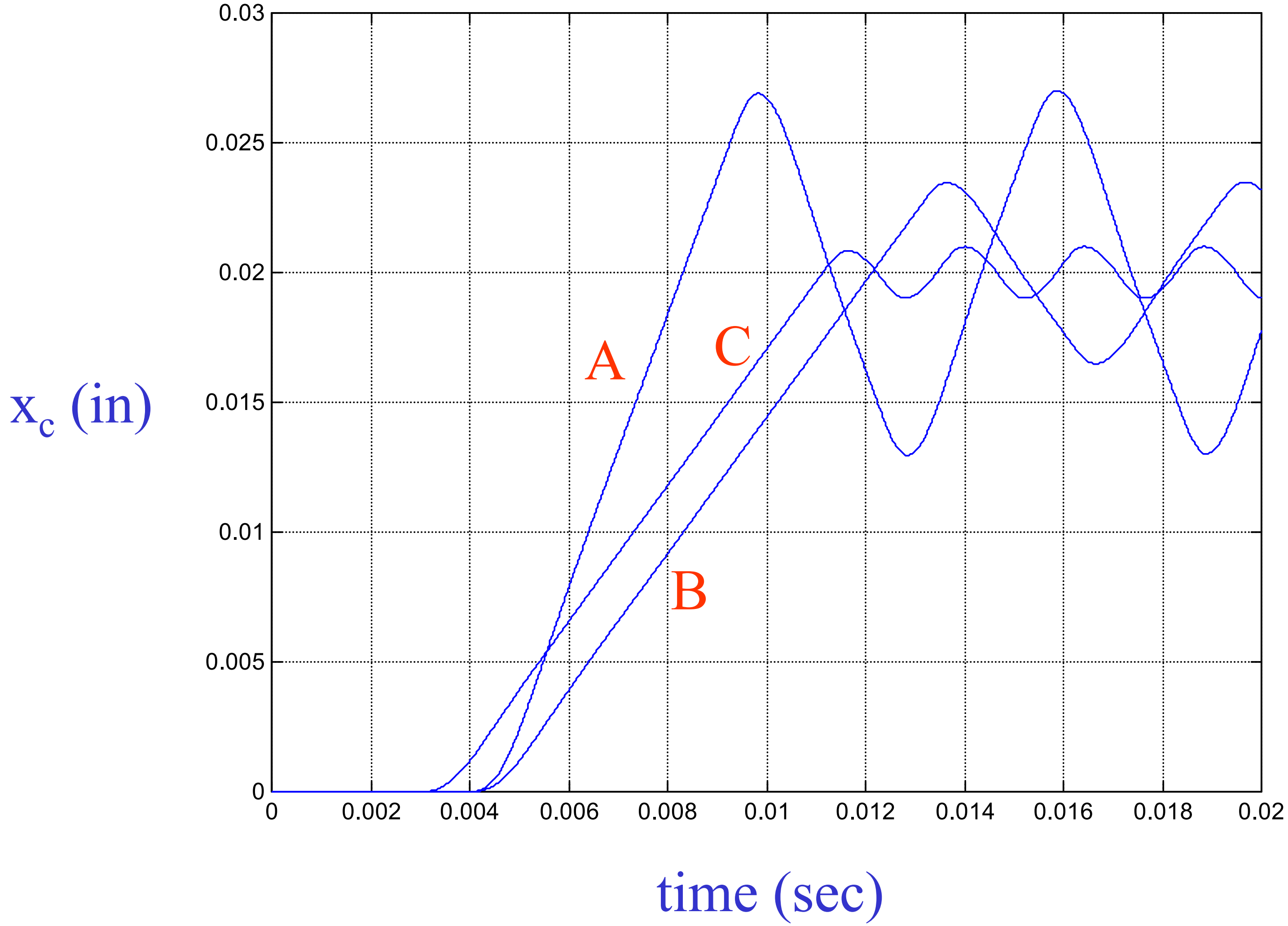


Simulink Block Diagram: Nonlinear Control with Time Delay

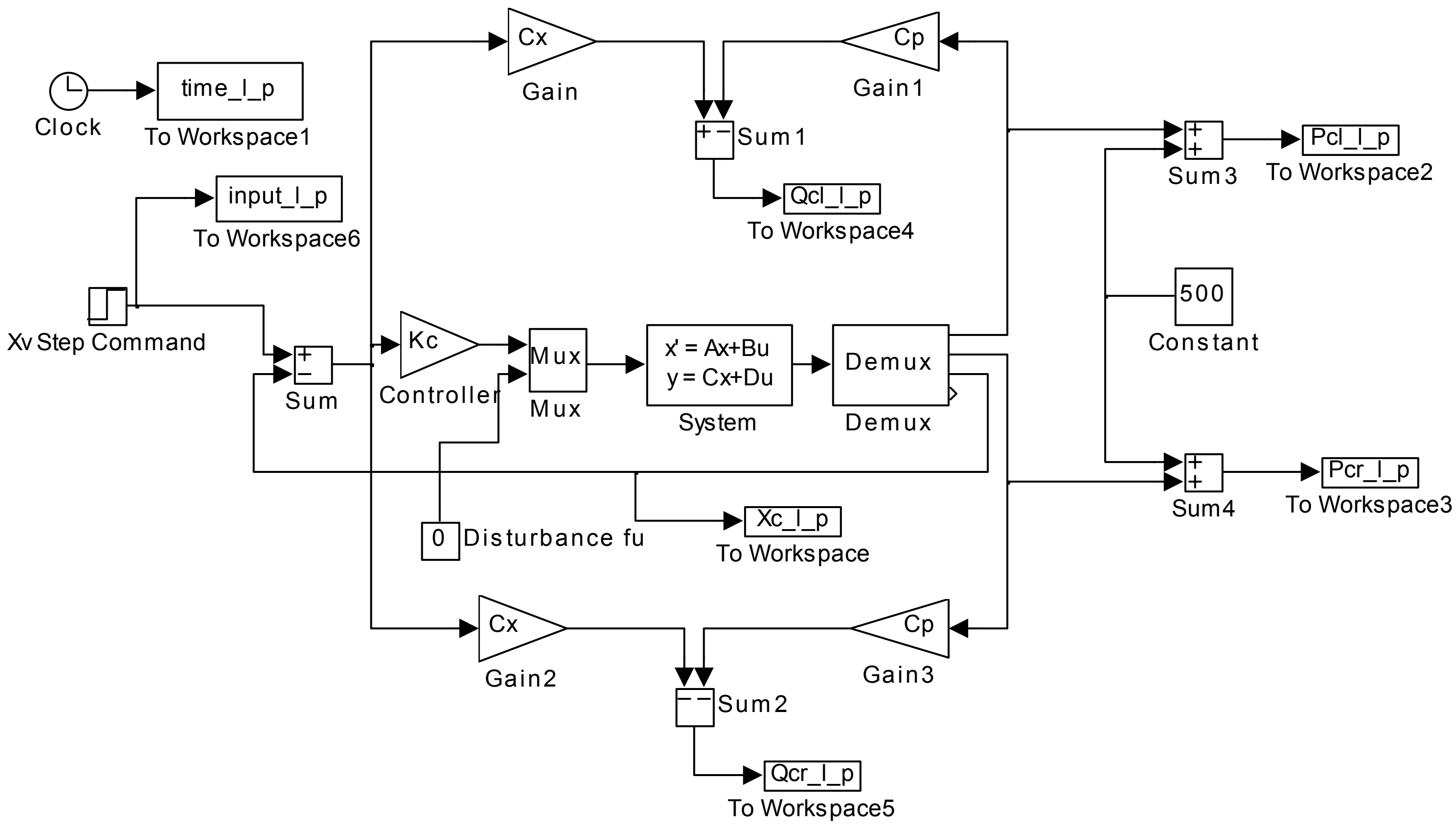


ELECTROHYDRAULIC VALVE-CONTROLLED SERVOMECHANISM (LINEAR)
with Nonlinear On-Off Controller and Time Delay

Curve A: Gain = 0.01, Delay = 0.001 sec
Curve B: Gain = 0.005, Delay = 0.001 sec
Curve C: Gain = 0.005, Delay = 0 sec



Simulink Block Diagram: Linear System with Proportional Control

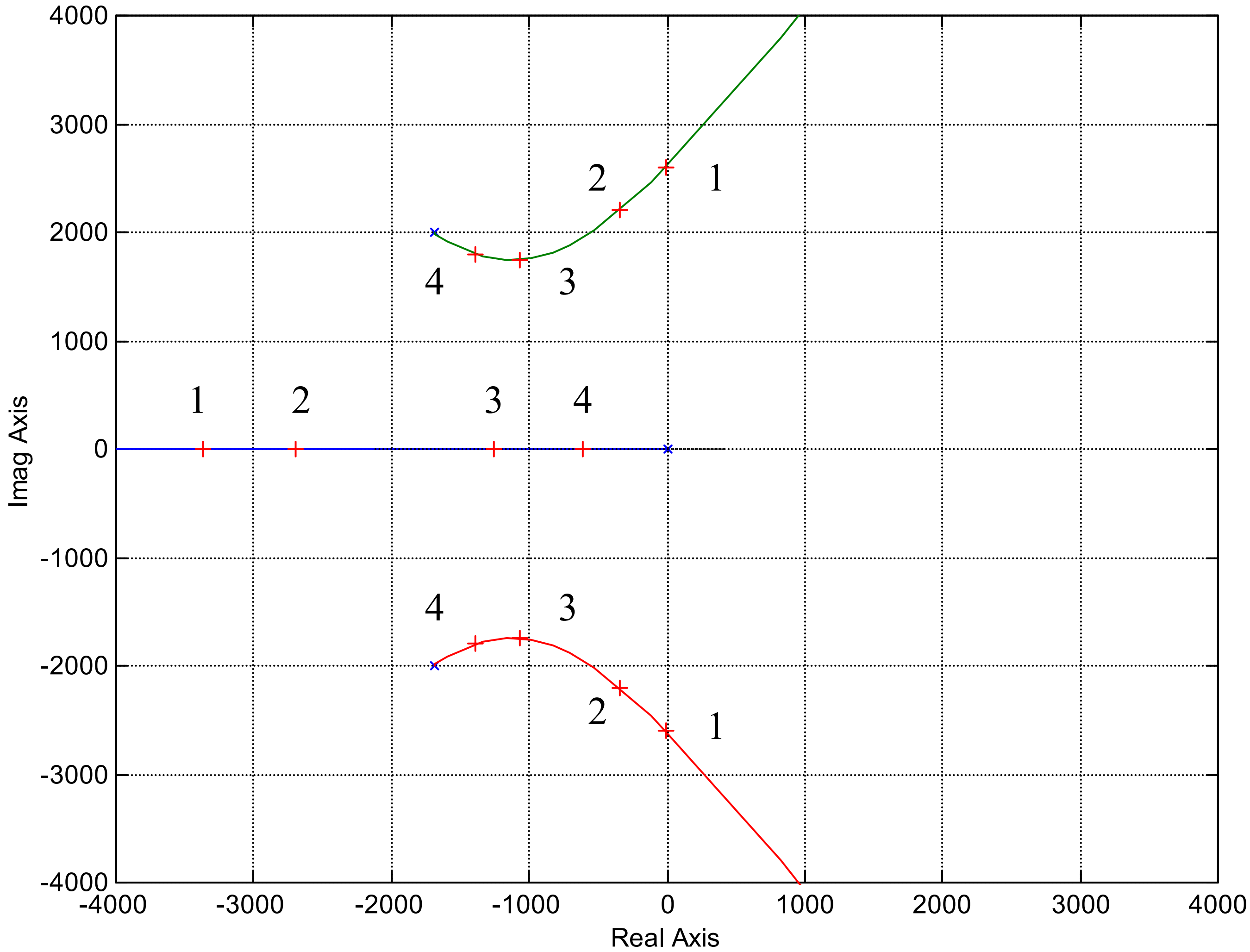


ELECTROHYDRAULIC VALVE-CONTROLLED SERVOMECHANISM (LINEAR)

With Proportional Control

Root Locus Plot

Pt. #1: $K_c = 6.46$ Pt. #2: $K_c = 3.70$
Pt. #3: $K_c = 1.36$ Pt. #4: $K_c = 1$



- Open-Loop Poles: 0, -1692 ± 1993i
- K_c =6.46

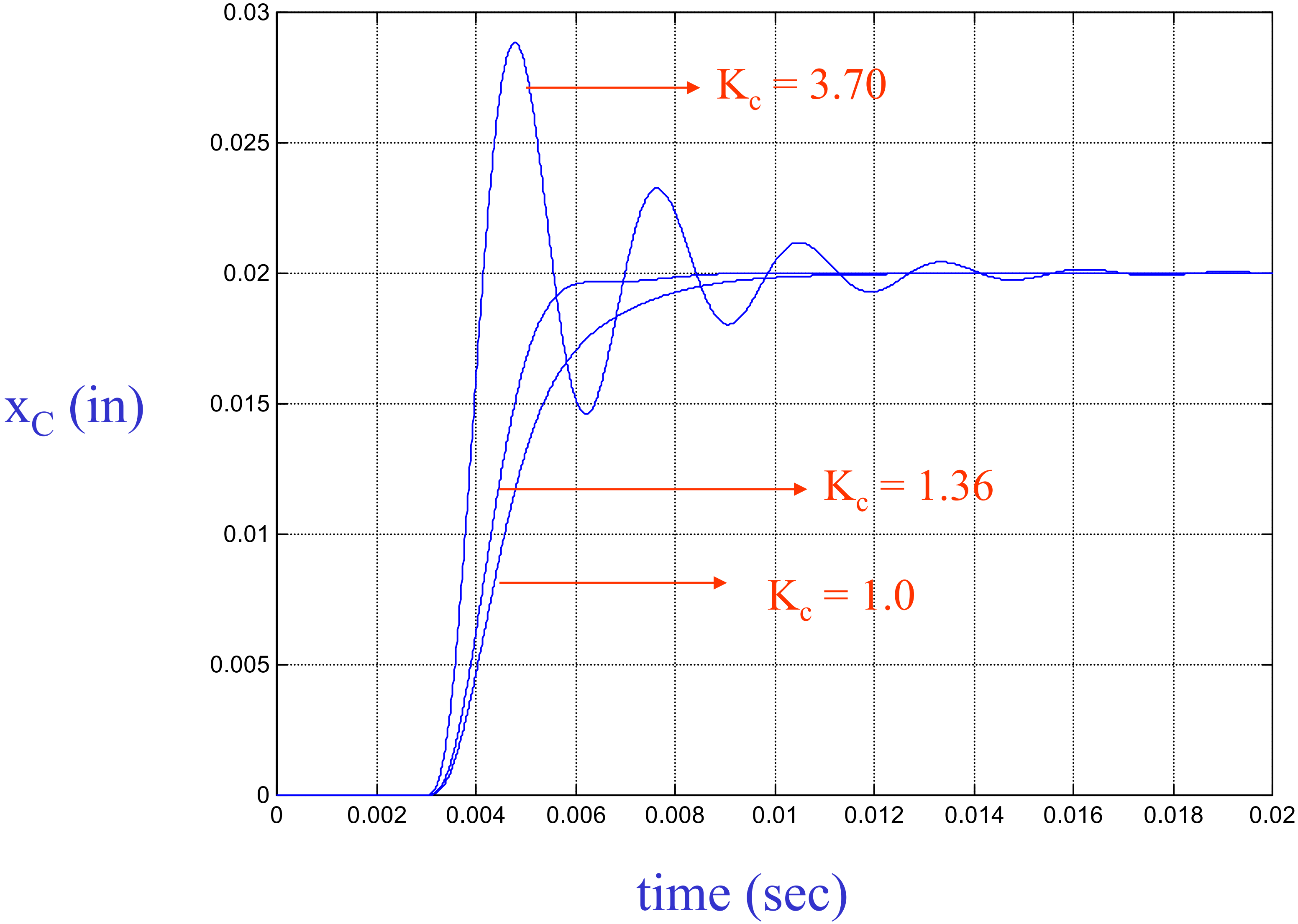
  Closed-Loop Poles: -3384, 0 ± 2614i
- K_c = 3.70

  Closed-loop Poles: -2677, -353 ± 2195i
- K_c = 1.36

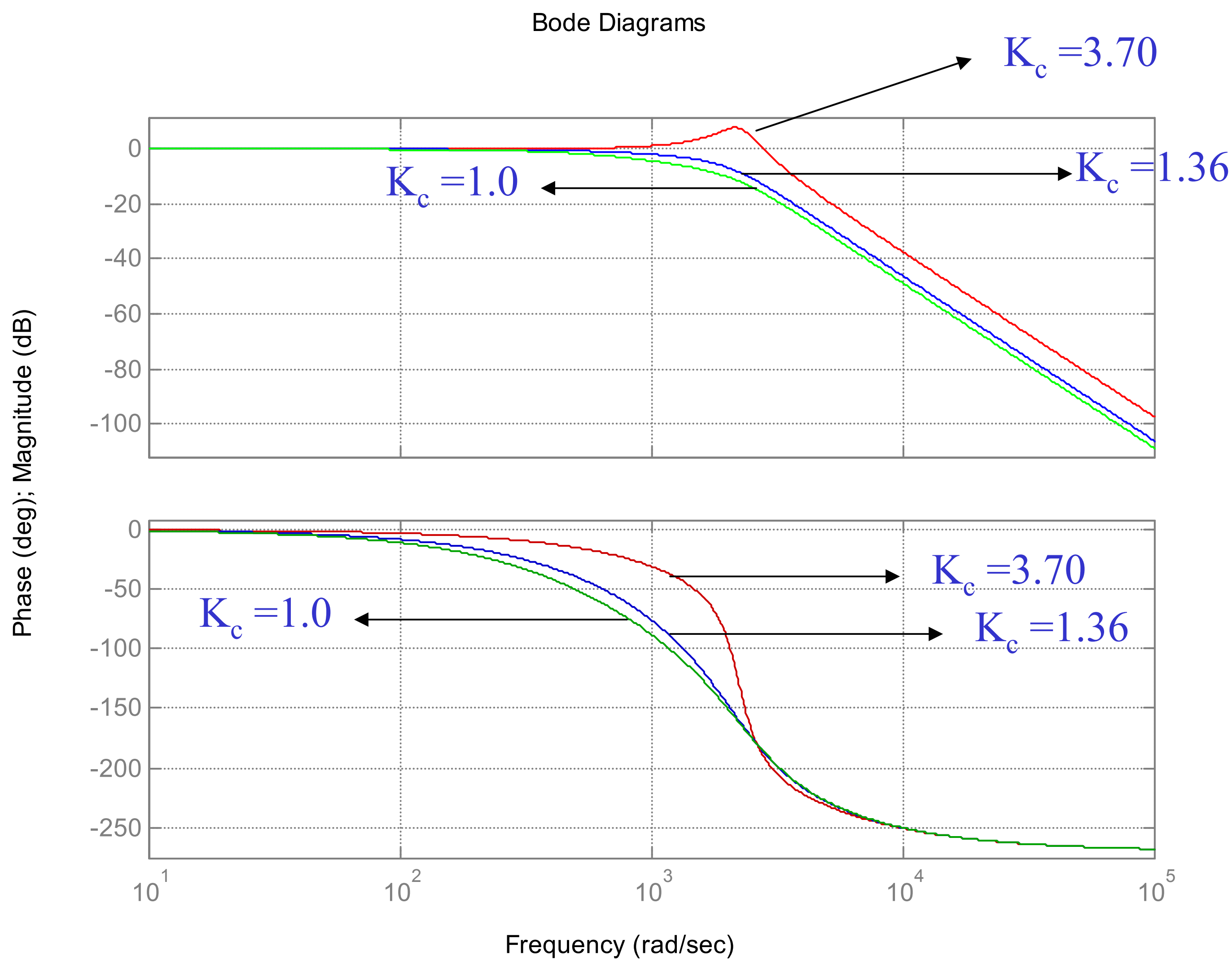
  Closed-Loop Poles: -1133, -1125 ± 1737i
- K_c = 1.0

  Closed-Loop Poles: -732, -1326 ± 1771i

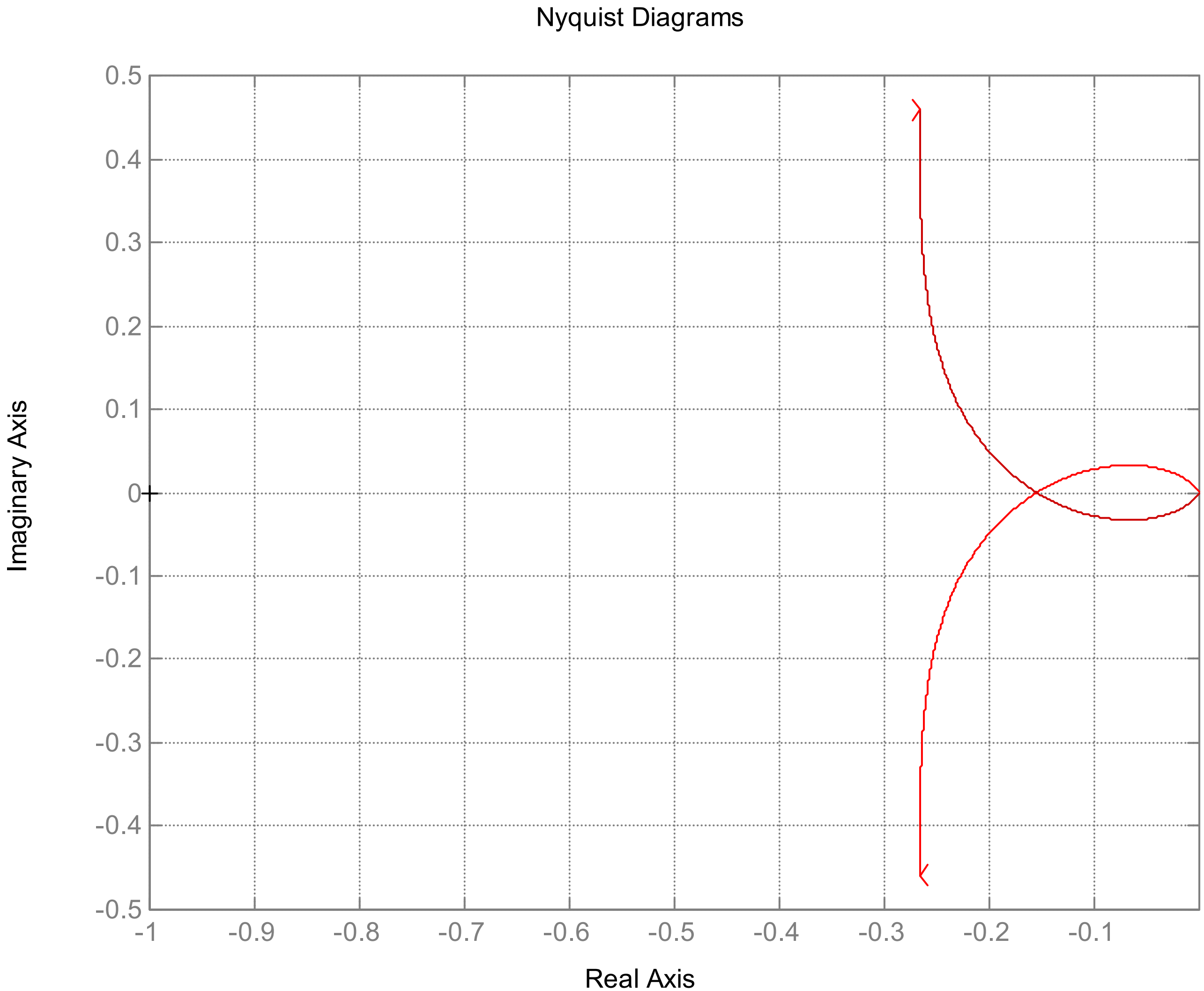
Closed-Loop Time Response (Step) Plots



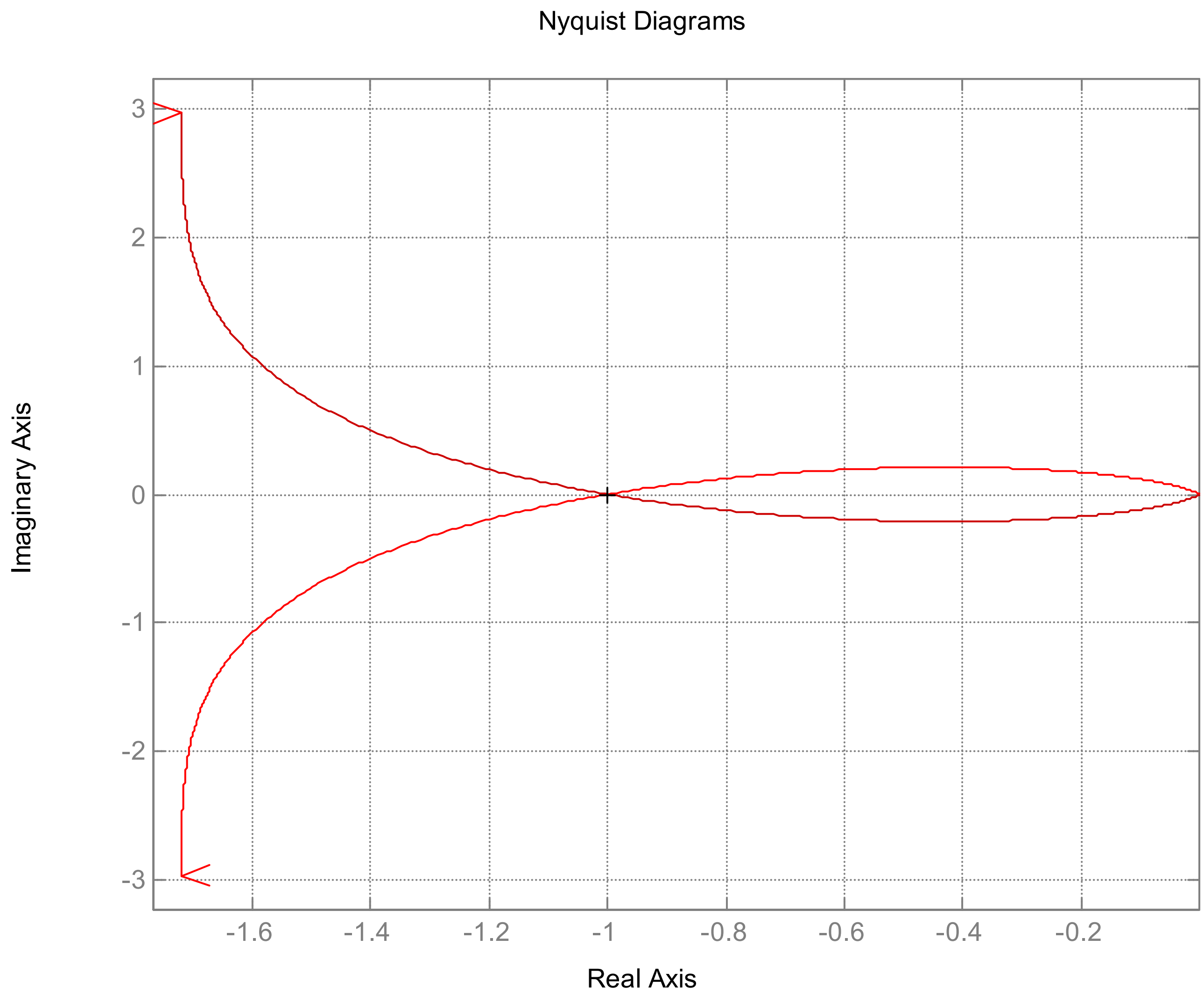
Closed-Loop Frequency Response Plots



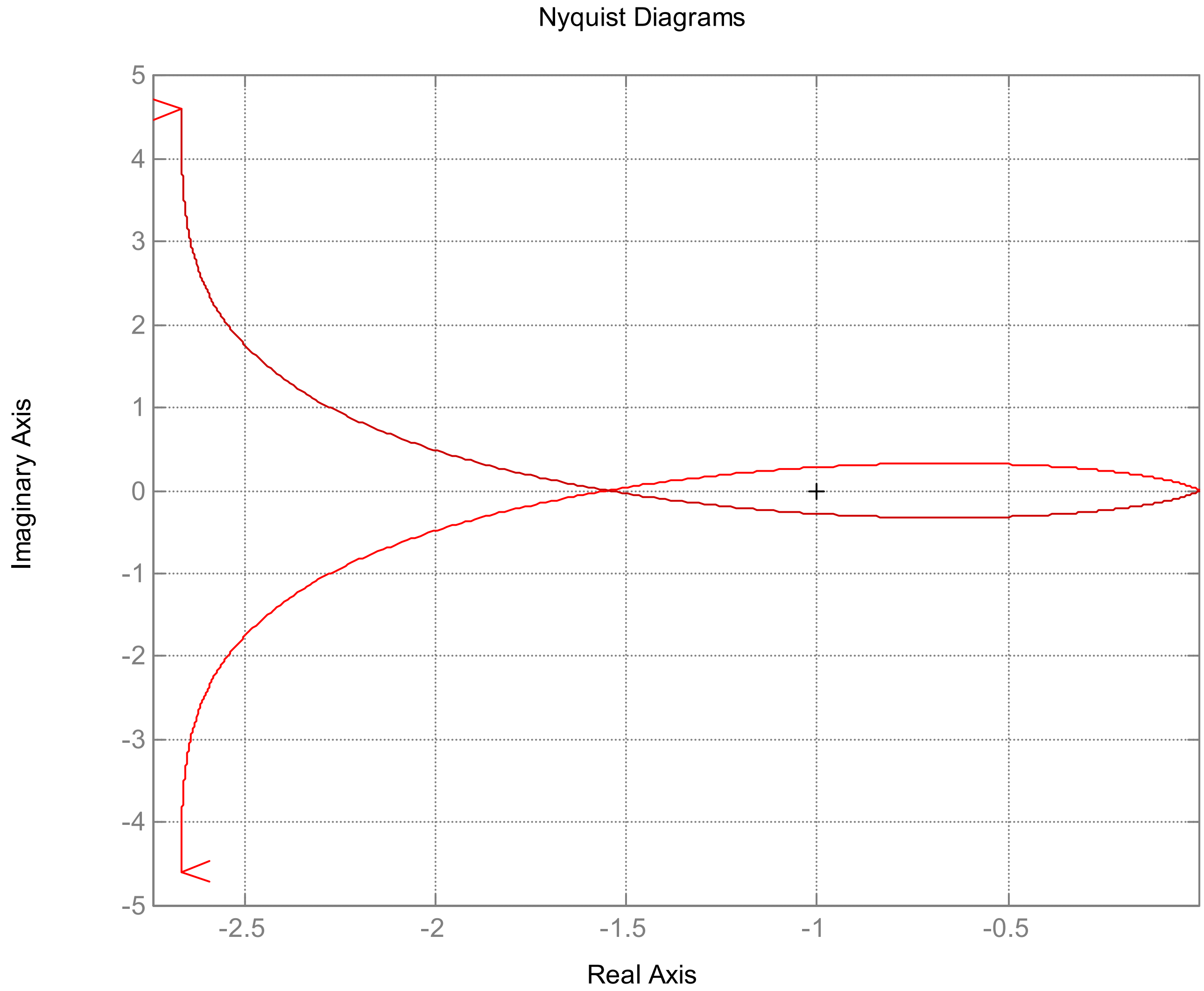
Nyquist Diagram: $K_c = 1.0$



Nyquist Diagram: $K_c = 6.46$



Nyquist Diagram: $K_c = 10.0$



Stability Considerations

Closed-Loop Transfer Function

$$\frac{x_C}{x_V} = \frac{K_c G(s)}{1 + K_c G(s)} = \frac{K_c K \omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_c K \omega_n^2}$$

$$K = \frac{2C_x A_p}{2A_p^2 + B\left(C_p + 2K_{pl}\right)}$$

$$\omega_n = \sqrt{\frac{\beta\left[2A_p^2 + B\left(C_p + 2K_{pl}\right)\right]}{MV_0}}$$

$$\zeta = \frac{B + \left(\frac{2\beta M}{V_0}\right)K_{pl} + \left(\frac{\beta M}{V_0}\right)C_p}{2\sqrt{\frac{\beta M}{V_0}\left[2A_p^2 + B\left(C_p + 2K_{pl}\right)\right]}}$$

- Neglect leakage ($K_{pl} = 0$) and consider the load as mainly inertia (B = 0, friction is ignored). The closed-loop transfer function becomes:

$$\frac{x_C}{x_V} = \frac{\dfrac{K_c C_x}{A_p}}{\dfrac{V_0 M}{2\beta A_p^2} s^3 + \dfrac{M C_p}{2 A_p^2} s^2 + s + \dfrac{K_c C_x}{A_p}}$$

- Since the bulk modulus β of the fluid is defined as:

$$\beta = -\frac{\partial P}{\partial V / V_0}$$

- The combined stiffness k_0 of the two columns of fluid is:

$$k_0 = \frac{2\beta A_p^2}{V_0}$$

- The valve stiffness k_v is defined as:

$$C_x = \left.\frac{\partial Q_v}{\partial x_v}\right|_{\text{operating point}}$$

$$C_p = -\left.\frac{\partial Q_v}{\partial p_c}\right|_{\text{operating point}}$$

$$k_v = 2A_p \frac{C_x}{C_p}$$

- The closed-loop transfer function can now be written as:

$$\frac{x_C}{x_V} = \frac{\dfrac{K_c C_x}{A_p}}{\dfrac{M}{k_0}s^3 + \dfrac{C_x M}{A_p k_v}s^2 + s + \dfrac{K_c C_x}{A_p}}$$

- Applying the Routh Stability Criterion to the characteristic equation of the closed-loop transfer function gives the relationship for stability as:

$$k_0 > k_v$$

- In other words, the stiffness of the oil column must be greater than the effective valve stiffness if stability is to be satisfactory.